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On the Upper Critical Dimension of Lattice Trees and Lattice Animals

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We give a rigorous proof of mean-field critical behavior for the susceptibility ($\gamma = 1/2$) and the correlation length ($\nu = 1/4$) for models of lattice trees and lattice animals in two cases: (i) for the usual model with trees or animals constructed from nearest-neighbor bonds, in sufficiently high dimensions, and (ii) for a class of "spread-out" or long-range models in which trees and animals are constructed from bonds of various lengths, above eight dimensions. This provides further evidence that for these models the upper critical dimension is equal to eight. The proof involves obtaining an infrared bound and showing that a certain "square diagram" is finite at the critical point, and uses an expansion related to the lace expansion for the self-avoiding walk.

KEY WORDS: Lattice animals; branched polymers; upper critical dimension; lace expansion; critical exponents; mean-field behavior.

1. THE MODELS AND RESULTS

In recent years there has been some progress in the rigorous study of critical phenomena for the self-avoiding walk and for percolation, above the upper critical dimension, where mean-field behavior takes over. The basic idea in this work is due to Brydges and Spencer,[1] who used their lace expansion to prove mean-field behavior (simple random walk scaling) for the weakly self-avoiding walk above four dimensions. A simplified convergence proof for the lace expansion was given in ref. 2, where it was proven that the mean square displacement of the strictly self-avoiding walk

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is linear in the number of steps, in sufficiently high dimensions. Further results for the strictly self-avoiding walk were obtained in refs. 3–5.

In ref. 6 it was shown how an expansion related to the lace expansion could be used to prove that the triangle condition[7] is satisfied in sufficiently high dimensions for independent nearest-neighbor percolation, and above six dimensions for a class of "spread-out" models of independent percolation. The triangle condition is known rigorously to imply that a number of percolation critical exponents take their mean-field (Bethe lattice) values.[7–9] Further results for percolation were obtained in ref. 10.

An important feature in this work (both for percolation and the self-avoiding walk) has been the proof of an infrared bound. Such a bound has played a major role in the rigorous analysis of the critical behavior of Ising and ϕ^4 models,[11–13] in particular in proving triviality above four dimensions. A general proof of the infrared bound is known for spin models which satisfy reflection positivity,[11] but no general proof of the infrared bound is known for the self-avoiding walk, percolation, or lattice animals. In fact, for percolation and lattice animals there is reason to believe that the infrared bound is violated below the upper critical dimension.[14–16]

In this paper we apply expansion methods to the study of lattice trees and lattice animals (also known as branched polymers) on the infinite d-dimensional hypercubic lattice $\mathbf{Z}^d$. By *lattice trees* we mean connected bond clusters without closed loops, and by *lattice animals* we mean connected bond clusters possibly with closed loops (see Section 1.1 for precise definitions). These models are of interest in polymer chemistry as well as in statistical physics. In addition, their statistics provides a natural problem in graph theory, which, like the statistics of the self-avoiding walk, has so far eluded a full solution by combinatoric or other methods.

A field-theoretic representation suggests that the upper critical dimension for lattice trees and lattice animals is eight,[15] i.e., above eight dimensions all critical exponents take their mean-field values, and below eight dimensions they do not. This suggestion is supported by the rigorous work of refs. 16–18. In particular, in refs. 17 and 18 it was shown that if the "square diagram" is finite at the critical point, as is believed to be the case for $d > 8$, then the critical exponent γ takes its mean-field value of 1/2. To our knowledge there is still no proof that mean-field values of critical exponents are incompatible with all $d < 8$. (For percolation it has been proven that mean-field critical exponents are incompatible with $d < 6$.[19,20]) A partial result in this direction is the proof in refs. 17, 18, and 21 that the critical exponent ν for the correlation length (assuming it exists) cannot take its mean-field value in less than four dimensions.

Our main result is a proof of mean-field power law behavior for the susceptibility ($\gamma = 1/2$) and correlation length of order two ($\nu = 1/4$) for lattice trees and lattice animals in two situations: (i) for systems involving only nearest-neighbor bonds, in sufficiently high dimensions, and (ii) for "spread-out" models involving both long and short bonds, above eight dimensions. On the basis of the hypothesis of universality, all critical exponents are believed to be the same for (i) and (ii), so the results for the spread-out models support $\gamma = 1/2$ and $\nu = 1/4$ also for the nearest-neighbor model above eight dimensions. This provides further evidence that the upper critical dimension is eight.

The proof uses an expansion quite similar to the lace expansion for the self-avoiding walk, which allows models (i) and (ii) to be treated as a small perturbation of corresponding simple random walk models. The small parameter responsible for convergence of the expansion is closely related to the square diagram (defined below). The analogue of this quantity for the usual nearest-neighbor simple random walk is small in very high dimensions, and is small also in any dimension above *eight* for a walk with variable-length (possibly quite long) steps, but it is infinite in either case if $d \leqslant 8$. It is not at all obvious that the square diagram is finite for (i) and (ii), let alone small, and one of our results is to prove that it is finite. We also prove a related infrared bound.

The mean-field bound $\gamma \geqslant 1/2$ was proven in ref. 16 for all dimensions. The opposite bound $\gamma \leqslant 1/2$ was shown in refs. 17 and 18 to be a consequence of the finiteness of the square diagram. Thus, it suffices to prove that the square diagram is finite to conclude that $\gamma = 1/2$. The power-law behavior of the correlation length, with exponent $\nu = 1/4$, follows from the arguments employed to prove the finiteness of the square diagram.

The exponent ν is often defined alternatively as the power at which the rate of exponential decay of the two-point function vanishes at the critical point. A rigorous proof that this exponent equals 1/4 for models (i) and (ii) can be obtained by combining our expansion with the method of ref. 10, in which analogous results were obtained for percolation. Formally, γ and ν are related respectively to the exponent θ for the number of n-bond trees or animals and to the exponent (also called ν) for the radius of gyration of n-bond trees or animals. Currently we have no proof of power law behavior for these microcanonical (as opposed to canonical) ensemble quantities.

1.1. The Models

We consider lattice trees and lattice animals on the infinite d-dimensional hypercubic lattice $\mathbf{Z}^d$. An element of $\mathbf{Z}^d$ is called a *site*, and an unordered pair $\{x, y\}$, where x, y are distinct sites, is called a *bond*. For now we restrict our attention to the nearest-neighbor bonds, for which

$\|x-y\|_2=1$; later more general bonds will be considered. A *lattice tree* is a connected set of *bonds* which has no closed loops. Although a tree T is defined as a set of bonds, we write $x \in T$ if x is an endpoint of some bond of T. We denote the number of *bonds* in a tree T by $|T|$. A *lattice animal* is a connected set of bonds, which may contain closed loops. We denote a typical lattice animal by A and the number of *bonds* in A by $|A|$. The number of *sites* in a tree or animal C will be denoted $\|C\|$. In general,

$$\|C\|-1 \leqslant |C| \leqslant \frac{\mathscr{X}}{2}\|C\| \tag{1.1}$$

where $\mathscr{X}$ is the coordination number ($\mathscr{X}=2d$ for the nearest-neighbor model, and more generally $\mathscr{X}$ is the maximum number of bonds which can emerge from one site). In fact, for trees the left inequality is always an equality.

The *two-point function* for trees or animals is defined, for sites $x, y \in \mathbf{Z}^d$, by

$$G_z(x, y) \equiv \sum_{T \ni x, y} z^{|T|}, \qquad G_z^a(x, y) \equiv \sum_{A \ni x, y} z^{|A|} \tag{1.2}$$

where z is a nonnegative parameter called the *activity*. The superscript a designates animals as opposed to trees. The *susceptibility* is defined by

$$\begin{aligned} \chi(z) &\equiv \sum_{x \in \mathbf{Z}^d} G_z(0, x) \equiv \sum_T{}' \|T\|^2 z^{|T|} \\ \chi^a(z) &\equiv \sum_{x \in \mathbf{Z}^d} G_z^a(0, x) \equiv \sum_A{}' \|A\|^2 z^{|A|} \end{aligned} \tag{1.3}$$

where $\sum'$ denotes a sum running over one tree T (or one animal A) in each equivalence class modulo translation.

We denote by a_n and a_n^a, respectively, the number of trees and animals containing n bonds, modulo translation. Thus, for example, $a_0=a_0^a=1$, $a_1=a_1^a=d$, and $a_2=a_2^a=d(2d-1)$. The existence (finite and nonzero) of the growth constants

$$\begin{aligned} \lambda &= \lim_{n \to \infty} a_n^{1/n} = \sup_{n \geqslant 1} a_n^{1/n} \\ \lambda_a &= \lim_{n \to \infty} (a_n^a)^{1/n} = \sup_{n \geqslant 1} (a_n^a)^{1/n} \end{aligned} \tag{1.4}$$

can be shown using subadditivity arguments (see ref. 22 for trees and ref. 23 for animals). In particular,

$$a_n \leqslant \lambda^n, \qquad a_n^a \leqslant \lambda_a^n \tag{1.5}$$

It was recently shown in ref. 24 that the growth constants σ and σ_a, defined by replacing a_n and a_n^a by the numbers s_n and s_n^a of *n-site* trees or animals, satisfy the strict inequality $\sigma < \sigma_a$. It is expected that

$$\begin{aligned} a_n &\sim n^{-\theta}\lambda^n, \qquad & a_n^a &\sim n^{-\theta}\lambda_a^n \\ s_n &\sim n^{-\theta}\sigma^n, \qquad & s_n^a &\sim n^{-\theta}\sigma_a^n \end{aligned} \tag{1.6}$$

with the same exponent θ in all cases. [The notation $\sim$ in (1.6) means that the left side is bounded uniformly above and below by positive multiples of the right side.] Note that (1.5) implies that $\theta \geqslant 0$. The mean-field (Bethe lattice) value of θ is 5/2, and this is expected to be correct on the lattice $\mathbf{Z}^d$ for $d > 8$.

The susceptibility has been proven to diverge at the *critical point* $z_c^{(a)} = \lambda_{(a)}^{-1}$; in fact,

$$\chi^{(a)}(z) \geqslant \text{const} \cdot (z_c^{(a)} - z)^{-1/2} \tag{1.7}$$

See refs. 16 and 18 and also Section 1.3 below. (Here the superscript a in parentheses is to be omitted or retained across the equation; we will use this convention throughout the paper to discuss trees and animals simultaneously.) On the basis of (1.6), it is expected that

$$\chi^{(a)}(z) \sim (z_c^{(a)} - z)^{-\gamma} \tag{1.8}$$

with $\gamma = 3 - \theta$. This can readily be seen for the analogue of $\chi^{(a)}$ defined by replacing $z^{|T|}$ and $z^{|A|}$ in (1.3) by $z^{\|T\|}$ and $z^{\|A\|}$; the same behavior is expected for the bond-weighted quantities. Note that (1.7) implies that $\gamma \geqslant 1/2$, and hence [assuming (1.6)] that $\theta \leqslant 5/2$. The mean-field value of γ is 1/2, which also is expected to be correct for $d > 8$. For $d < 8$, γ and all other critical exponents are expected to be dimension dependent.

For $z < z_c^{(a)}$, we define the *correlation length of order two* $\xi^{(a)}(z)$ by

$$\xi^{(a)}(z) = \left[\frac{\sum_x |x|^2 G_z^{(a)}(0, x)}{\sum_x G_z^{(a)}(0, x)} \right]^{1/2} \tag{1.9}$$

It is expected that the correlation length diverges at the critical point via a power law:

$$\xi^{(a)}(z) \sim (z_c^{(a)} - z)^{-\nu} \tag{1.10}$$

The mean-field value of ν is 1/4 (see, for example, ref. 16) and it is expected that $\nu = 1/4$ also for $\mathbf{Z}^d$ when $d > 8$. An alternate correlation length is $\bar{\xi}^{(a)}(z)$ (the inverse mass gap), defined by

$$\bar{\xi}^{(a)}(z)^{-1} \equiv -\limsup_{n \to \infty} \frac{\log G_z^{(a)}(0, n(1, 0, ..., 0))}{n}$$

It is expected that as $z \nearrow z_c^{(a)}$,

$$\xi^{(a)}(z) \sim (z_c^{(a)} - z)^{-\bar{\nu}}$$

and that $\bar{\nu} = \nu$. Formally ν also describes the behavior of the radius of gyration $R^{(a)}$ of n-bond trees or animals:

$$R^{(a)}(n)^2 \equiv \frac{\sum_{C \ni 0: |C| = n} \sum_{x \in C} |x - \bar{x}_C|^2}{\sum_{C \ni 0: |C| = n} \sum_{x \in C} 1} \sim n^{2\nu} \tag{1.11}$$

where C denotes a tree for R and an animal for R^a, and $\bar{x}_C$ is the center of mass of C. In fact, $(\xi^{(a)})^2$ is exactly twice the average squared radius of gyration:

$$(\xi^{(a)})^2 = 2 \frac{\sum_n z^n \sum_{C \ni 0, |C| = n} \|C\| \cdot R^{(a)}(n)^2}{\sum_n z^n \sum_{C \ni 0, |C| = n} \|C\|}$$

At the critical point the two-point function is expected to decay at large distances according to a power law

$$G_{z_c^{(a)}}^{(a)}(0, x) \sim |x|^{-(d-2+\eta)} \tag{1.12}$$

The critical exponents γ, ν, η are believed on the basis of scaling theory to satisfy the identity $\gamma = (2 - \eta)\nu$. For $d > 8$ it is expected that $\eta = 0$, while for $2 \leqslant d < 8$ it is expected that η is negative.[15,16] In particular, for $d = 3$, Parisi–Sourlas dimensional reduction[25] predicts that $\eta = -1$.[16] In terms of the Fourier transform

$$\hat{G}_z^{(a)}(k) \equiv \sum_{x \in \mathbf{Z}^d} G_z^{(a)}(0, x)\, e^{ikx}, \qquad k \in [-\pi, \pi]^d$$

(1.12) is consistent with the behavior

$$\hat{G}_{z_c^{(a)}}^{(a)}(k) \sim \frac{1}{k^{2-\eta}}$$

for k near 0. Thus, we expect that for $d > 8$, $\hat{G}$ should obey the *infrared bound*

$$\hat{G}_{z_c^{(a)}}^{(a)}(k) \leqslant \frac{C}{k^2} \tag{1.13}$$

If the infrared bound (1.13) is obeyed, then it would follow that for $d > 8$ the *square diagram*

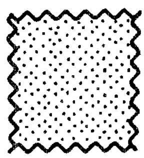
$$\begin{aligned}\square^{(a)}(z) &\equiv \sum_{w, x, y \in \mathbf{Z}^d} G_z^{(a)}(0, w)\, G_z^{(a)}(w, x)\, G_z^{(a)}(x, y)\, G_z^{(a)}(y, 0) \\ &= \int_{[-\pi, \pi]^d} \frac{d^d k}{(2\pi)^d} [\hat{G}_z^{(a)}(k)]^4 \end{aligned} \tag{1.14}$$

would be finite at the critical point. It is known that if $\square^{(a)}(z_c^{(a)}) < \infty$, then (1.8) holds with $\gamma = 1/2$, in the sense that

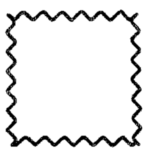
$$c_1(z_c^{(a)} - z)^{-1/2} \leqslant \chi^{(a)}(z) \leqslant c_2(z_c^{(a)} - z)^{-1/2} \tag{1.15}$$

for some positive constants $c_1 < c_2$. In fact, the lower bound holds quite generally,[16,18] while the upper bound follows from the finiteness of the square diagram;[17,18] see Section 1.3. (References 16–18 explicitly treat only the nearest-neighbor model, but the methods apply equally to the spread-out models.)

The square diagram gets its name from the following Feynman diagram notation, which we use repeatedly. We use $0 \sim\sim\sim x$ to denote the two-point function or *propagator* $G_z^{(a)}(0, x)$. In any diagram we use the convention that each line denotes a propagator, unlabeled vertices are summed over the lattice, and in a diagram with all vertices unlabeled one vertex is fixed at the origin. Thus,

$\square^{(a)}(z) =$ 

A shaded loop involves an unconstrained sum over vertices, whereas in an unshaded loop we use the convention that the summation over vertices is restricted to omit the coincidence of all vertices on the loop. For example,

$S_z^{(a)} \equiv \square^{(a)}(z) - G_z^{(a)}(0, 0)^4 =$  (1.16)

In this paper we prove mean-field behavior for $\chi^{(a)}(z)$ and $\xi^{(a)}(z)$ in sufficiently high dimensions. The proof uses an expansion related to the lace expansion for the self-avoiding walk, whose convergence is assured by taking d sufficiently large. In order to obtain mean-field behavior right down to $d = 8^+$, we consider a model of "spread-out" trees and animals which introduces a small parameter to replace d^{-1}. Similar models were considered for percolation in ref. 6. In the spread-out model, trees and animals can be built from any bonds $\{x, y\}$, $x, y \in \mathbf{Z}^d$, $x \neq y$, for which $y - x \in \Omega \backslash \{0\}$, where $\Omega \subset \mathbf{R}^d$ is a given compact set with finitely many connected components and positive Lebesgue measure. In particular, we consider Ω of the form $\Omega_L = \{x\colon x/L \in \Omega_1\}$, where L is large and Ω_1 is a fixed compact subset of $\mathbf{R}^d$ which is $\mathbf{Z}^d$-invariant (i.e., invariant under reflections

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in the coordinate hyperplanes and rotations by $\pi/2$ about the coordinate axes).

For the spread-out models we define the two-point function $G_z^{(a),L}(x, y)$ by replacing the sums over nearest-neighbor trees and animals in (1.2) by sums over trees and animals built from bonds $\{x, y\}$ for which $y-x \in \Omega_L \backslash \{0\}$. Then we define $\chi^{(a),L}(z)$, $z_c^{(a),L}$, $\xi^{(a),L}$, and $\square^{(a),L}$ by replacing $G_z^{(a)}$ by $G_z^{(a),L}$ in (1.3), (1.9), and (1.14). Similarly, we use $a_n^{(a),L}$ to denote the number of spread-out trees or animals containing exactly n bonds.

1.2. The Results

In this paper we prove the following theorems:

Theorem 1.1. For the nearest-neighbor models of lattice trees and lattice animals there is a $d_0 > 8$ such that for $d \geqslant d_0$:

(a) The infrared bound $0 \leqslant \hat{G}_z^{(a)}(k) \leqslant c/k^2$ is satisfied uniformly in $z < z_c^{(a)}$. The square diagram $\square^{(a)}(z_c^{(a)})$ is finite, and hence $\gamma = 1/2$ in the sense that (1.15) is satisfied.

(b) $\nu = 1/4$ in the sense that there are constants $0 < c_3 < c_4 < \infty$ such that for z less than but near $z_c^{(a)}$,

$$c_3(z_c^{(a)} - z)^{-1/4} \leqslant \xi^{(a)}(z) \leqslant c_4(z_c^{(a)} - z)^{-1/4}$$

Theorem 1.2. Let Ω_1 be a compact $\mathbf{Z}^d$-invariant subset of $\mathbf{R}^d$ with finitely many connected components and positive Lebesgue measure. Then, for any $d > 8$ there is an $L_0 = L_0(d, \Omega_1)$ such that for $L \geqslant L_0$ the spread-out models of lattice trees and lattice animals defined by Ω_1 obey (a) and (b) of Theorem 1.1.

On the basis of the hypothesis of universality, all critical exponents for the spread-out model are expected to be the same as for the nearest-neighbor model, independent of Ω_1. In particular, Theorem 1.2 *suggests* that $\gamma = 1/2$ and $\nu = 1/4$ for the nearest-neighbor model above eight dimensions. Theorem 1.2 shows that above eight dimensions γ and ν are independent of Ω_1 if L is sufficiently large, which supports the hypothesis of universality.

The proofs of Theorems 1.1 and 1.2 are essentially the same, as are the proofs for trees and animals. As mentioned in Section 1.1 and further explained in Section 1.3, to show that $\gamma = 1/2$, it suffices to show that the square diagram is finite at the critical point. For this, we first obtain a uniform bound on the square diagram below the critical point. It will then follow from the monotone convergence theorem that the square diagram is also finite at the critical point. The infrared bound is proved simultaneously. The basic mechanism in the proof was used in ref. 2, where in particular it was shown that the bubble diagram for the self-avoiding walk is finite at the critical point. It involves an easy continuity argument for the square and the weighted triangle (defined in Section 2.3) as functions of the activity z, together with a more difficult argument which uses the lace expansion to show that when d or L is sufficiently large, if $S(z)$ $(z < z_c)$ is less than 4ε, then in fact it is less than 3ε, where ε is proportional to d^{-1} or an inverse power of L. In the course of the proof it can be seen that

$$\xi^{(a)}(z)^2 \sim \chi^{(a)}(z)$$

which then gives (b).

Throughout the paper we shall concentrate on trees, commenting briefly on the modifications needed to treat animals. The remainder of the paper is organized as follows. In Section 1.3, we discuss the relation between (1.15) and the finiteness of the square diagram in more detail. In Section 2, we recall the lace expansion and show how it can be applied to trees and animals, both nearest-neighbor and spread-out. In Section 3 the proof of Theorem 1.2 is given. The proof of Theorem 1.1 is almost identical, and the details are not repeated. Finally, in an Appendix we collect some Gaussian (simple random walk) bounds which are needed in Section 3.

1.3. The Skeleton Inequalities

The upper bound of (1.15), which implies that $\gamma \leqslant 1/2$, was shown in refs. 17 and 18 to follow from the finiteness of the square diagram, while the lower bound of (1.15), which implies that $\gamma \geqslant 1/2$, holds in general.[16,18] In this paper we prove that under the hypotheses of Theorem 1.2, $S_{z_c}^{(a)} \leqslant 1$ and $G_{z_c}^{(a)}(0,0) \leqslant 4$. (Similar bounds hold under the hypotheses of Theorem 1.1.) This is a stronger statement than $\square(z_c) < \infty$, and it is actually less involved to prove that the stronger statement implies that $\gamma \leqslant 1/2$ than it is to prove that finiteness of the square diagram implies $\gamma \leqslant 1/2$. To make this paper more self-contained, we briefly explain the argument in this section.

The argument follows the basic strategy presented in refs. 16 and 18. To simplify the notation, we omit labels (a), L, and z, and use C to denote either a tree or an animal. By definition,

$$z \frac{d\chi(z)}{dz} = \sum_x \sum_{C \ni 0,x} |C| \, z^{|C|}$$

By (1.1),

$$\sum_{C \ni 0,x} (\|C\| - 1) \, z^{|C|} \leqslant \sum_{C \ni 0,x} |C| \, z^{|C|} \leqslant \sum_{C \ni 0,x} \frac{\mathscr{Z}}{2} \|C\| \, z^{|C|} \qquad (1.17)$$

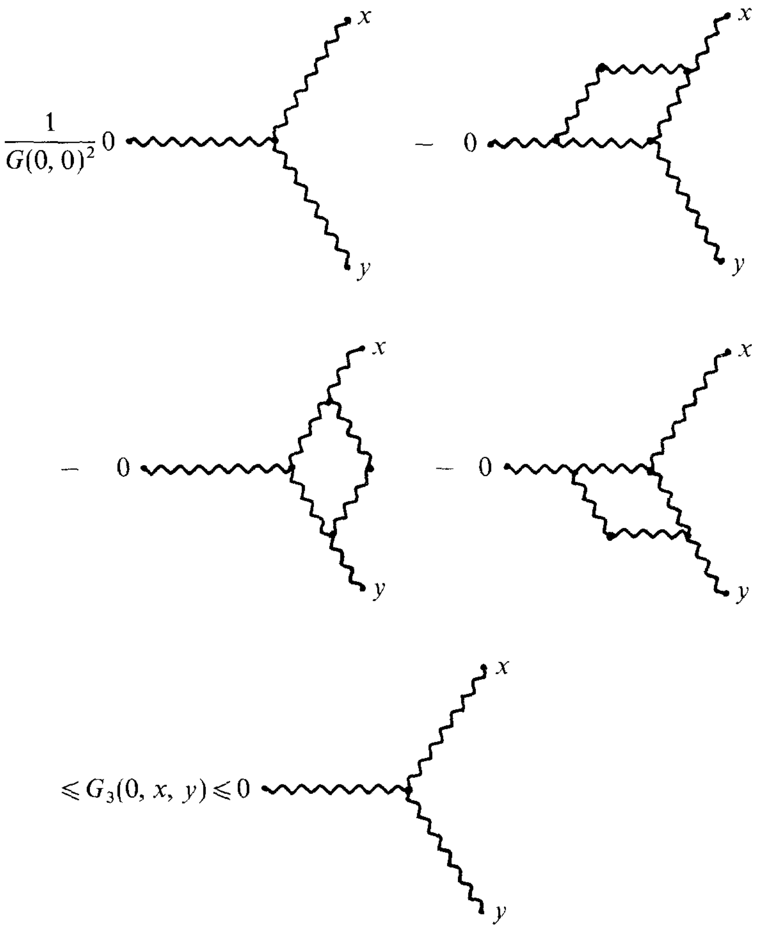
Let

$$G_3(0, x, y) \equiv \sum_{C \ni 0, x, y} z^{|C|}$$

Then from (1.17) we have

$$\sum_{x, y} G_3(0, x, y) - \chi(z) \leqslant z \frac{d\chi(z)}{dz} \leqslant \frac{\mathscr{L}}{2} \sum_{x, y} G_3(0, x, y)$$

As we briefly explain below, $G_3(0, x, y)$ can be bounded above and below using the first- and second-order skeleton inequalities [ref. 16, Eqs. (5.63), (5.69); ref. 18, Eqs. (5.8), (5.14)]:



(1.18)

[In fact, (5.14) of ref. 18 is not correct for animals as stated: the factor $e^{2\beta}$ on the right side should be replaced by $G_\beta(0, 0)^{-2}$.] The loops in (1.18) are unshaded, i.e., the squares on the left side involve the quantity S given in (1.16) rather than $\square$. As a result,

$$\chi(z)^3\,[G(0,0)^{-2}-3S]-\chi(z)\leqslant z\frac{d\chi(z)}{dz}\leqslant\frac{\mathscr{X}}{2}\chi(z)^3 \tag{1.19}$$

If $S<1/[3G(0,0)^2]$ uniformly in $z\leqslant z_c$, then (1.19) provides uniform positive upper and lower bounds on $|d(\chi(z)^{-2})/dz|$, which upon integration yields (1.15). In this paper we in fact prove that under the hypotheses of Theorem 1.2, $G(0,0)\leqslant 4$ and $S\ll 1$ uniformly in $z\leqslant z_c$ (a similar result holds for the nearest-neighbor model); see Section 3.1.

We now briefly sketch the derivation of the skeleton inequalities (1.18). For trees the situation is very similar to that described in some detail in ref. 16. The derivation of the skeleton inequalities for animals is more delicate, and we discuss only this case.

For the first-order skeleton inequality [the upper bound of (1.18)], we first note that for any animal A contributing to $G_3(x_1,x_2,x_3)$, we can find a site $w\in A$ (possibly equal to one of x_1, x_2, x_3) and paths in A (possibly consisting of the single site w) from x_1 to w, from w to x_2, and from w to x_3, with these three paths intersecting only at w. The choice of w and the three paths will in general not be unique, but we can impose an order on the set consisting of such sets of three paths which will allow us to associate to each animal A a unique site w and triple of paths, as above. Next we use some algorithm (for example, see the proof of Lemma 2.1) to decompose A into three animals A_1, A_2, A_3 containing the three paths, with $A=A_1\cup A_2\cup A_3$ (as sets of *bonds*), and $A_i\cap A_j=\varnothing$, $i\neq j$, again as sets of bonds. Since $A=A_1\cup A_2\cup A_3$, different A's have different decompositions A_1, A_2, A_3, and hence the upper bound of (1.18) can be obtained by over-counting:

$$G_3(x_1,x_2,x_3)\leqslant\sum_w\sum_{A_i\ni w,x_i} z^{|A_1|+|A_2|+|A_3|}=\sum_w G(x_1,w)\,G(w,x_2)\,G(w,x_3)$$

Similar upper bounds will be needed in Section 2.2.

For the second-order skeleton inequality [the lower bound of (1.18)], we bound the three-point function $G_3(x_1,x_2,x_3)$ below by summing over only those $A\ni x_1$, x_2, x_3 for which there is a unique pivotal site w, i.e., a site such that the removal of all bonds in A having endpoint w will disconnect x_i from x_j $(i\neq j)$. For such A there are three animals A_1, A_2, A_3 such that (1) $A_i\cap A_j=\{w\}$, $i\neq j$, as sets of *sites*, (2) $A_i\ni x_i$, w, (3) $A=A_1\cup A_2\cup A_3$, as sets of *bonds*. Although the animals A_1, A_2, A_3 are in general not uniquely determined, the only ambiguity in the composition of the A_i is due to possible "branches" in A, emanating from w, which do not include one of the x_i. We remove this ambiguity by assigning all such branches to A_1; the animals A_2 and A_3 are then "trimmed at w" in the sense that any bond emanating from w in A_2 or A_3 is the first step in a self-avoiding walk in A_j from w to x_j. This gives the lower bound

$$G_3(x_1, x_2, x_3) \geqslant \sum_w \sum_{A_i \ni x_i, w} z^{|A_1| + |A_2| + |A_3|} I[A_2 \text{ and } A_3 \text{ are trimmed at } w] \times I[A_i \cap A_j = \{w\}, i \neq j]$$

where I denotes the indicator function.

Now by inclusion–exclusion we have

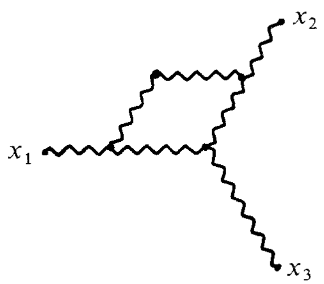
$$I[A_i \cap A_j = \{w\}, i \neq j] \geqslant 1 - I[A_1 \cap A_2 \supset \{w\}] - I[A_2 \cap A_3 \supset \{w\}] - I[A_1 \cap A_3 \supset \{w\}]$$

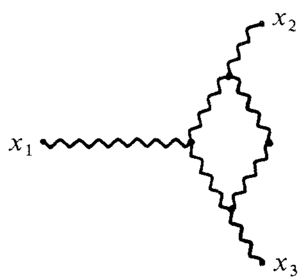
where $\supset$ denotes strict containment. Using

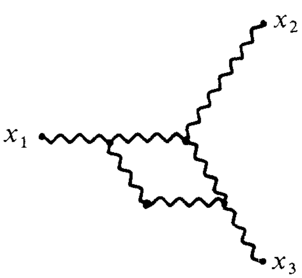
$$\sum_{\substack{A_1 \ni w, x_1 \\ A_2 \ni w, x_2}} I[A_1 \cap A_2 \supset \{w\}] \, z^{|A_1| + |A_2|} \leqslant \sum_{y \neq w} G_3(w, x_1, y) \, G_3(w, x_2, y)$$

together with the first-order skeleton inequality leads to the lower bound

$$G_3(x_1, x_2, x_3) \geqslant \sum_w G(x_1, w) \, G'(w, x_2) \, G'(w, x_3)$$

$-$ 

$-$ 

$-$ 

where

$$G'(w, x) = \sum_{A \ni w, x} z^{|A|} I[A \text{ is trimmed at } w]$$

The lower bound of (1.18) now follows from the fact that $G'(w, x) \geq G(0, 0)^{-1} G(w, x)$.

The skeleton inequalities (1.18) can be derived as above for both trees and animals when the two-point function is defined as in (1.2) using the weight $z^{|C|}$. These inequalities are, however, problematic for animals if the two-point function is defined using the weight $z^{\|A\|}$, and it is because of this and similar difficulties with the latter weight which would occur in Section 2.2.2 that we have used the weight $z^{|A|}$. More precisely, our analysis goes through if we either (i) define animals as sets of *bonds* and use the weight $z^{|A|}$, or (ii) define animals as sets of *sites* and use the weight $z^{\|A\|}$. The latter case was considered in ref. 18. However, if we define animals as sets of *bonds* and use the weight $z^{\|A\|}$, then we encounter difficulties arising from the fact that there can be a large number of different bond animals with the same set of endpoints of their bonds (and thus with the same weight $z^{\|A\|}$).

2. THE EXPANSION

2.1. Derivation of the Expansion

In this section we derive the expansion used in the proof of Theorems 1.1 and 1.2. The expansion is closely related to the lace expansion for the self-avoiding walk.[1] In Section 2.1.1 we consider in detail the case of lattice trees, nearest-neighbor or spread-out, and then indicate in Section 2.1.2 how the expansion is modified for lattice animals. We omit the label L indicating quantities for the spread-out trees or animals. Bounds on each term of the expansion are obtained in Section 2.2.

2.1.1. The Expansion for Trees. Given two distinct sites x, y and a tree $T \ni x, y$ (nearest-neighbor or spread-out), the *backbone* $\beta_T(x, y)$ of T is defined to be the unique path, consisting of bonds of T, which joins x to y. Usually x and y are understood and we write simply β_T for $\beta_T(x, y)$. Sites in the backbone are labeled consecutively from x to y, beginning with $\beta_T(0) = x$ and ending at (say) $\beta_T(n) = y$. Removal of the bonds in the backbone disconnects the tree into $n+1$ mutually nonintersecting trees $R_0, \ldots, R_n$, which we refer to as *ribs*. This decomposition is shown in Fig. 1. Ribs are defined as sets of *bonds*, but as for trees and animals, we write $R \ni x$ if a site x is an endpoint of some bond of the rib R.

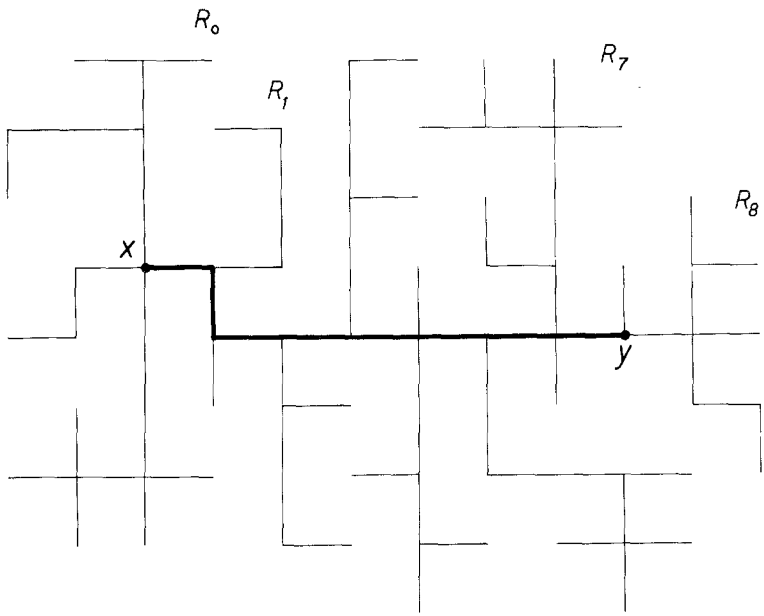


Fig. 1. Decomposition of a tree T containing sites x and y into its backbone $\beta_T(x, y)$ and ribs $R_0,..., R_n$. The backbone is indicated by a bold line.

Given a set $\mathbf{R} = \{R_0,..., R_n\}$ of $n+1$ trees R_j, we define

$$\mathcal{U}_{st}(\mathbf{R}) = \begin{cases} -1 & \text{if } R_s \text{ and } R_t \text{ share a common site} \\ 0 & \text{if } R_s \text{ and } R_t \text{ share no common site} \end{cases} \tag{2.1}$$

Then the two-point function can be written

$$G_z(0, x) = \sum_{\omega: 0 \to x} z^{|\omega|} \left[\prod_{i=0}^{|\omega|} \sum_{R_i \ni \omega(i)} z^{|R_i|} \right] \prod_{0 \leqslant s < t \leqslant |\omega|} [1 + \mathcal{U}_{st}(\mathbf{R})] \tag{2.2}$$

where each sum over R_i is a sum over trees and $\mathbf{R} = (R_0,..., R_{|\omega|})$. The sum over ω is the sum over all simple (possibly self-intersecting) walks from 0 to x, although walks which do have self-intersections give zero contribution to (2.2). For nearest-neighbor trees, ω takes nearest-neighbor steps, while for spread-out trees, ω takes steps (u, v) with $v - u \in \Omega_L \backslash \{0\}$. As usual, $|R_i|$ and $|\omega|$ denote the number of *bonds* in R_i and ω, respectively.

To describe the expansion, it is necessary to first introduce some terminology. Given an interval $I = [a, b]$ of nonnegative integers, we refer to a pair st of elements of I as an *edge*. A set of edges is called a *graph*. A graph Γ is said to be *connected* if both a and b are endpoints of edges in

Γ, and if, in addition, for any $c \in (a, b)$, there are $s, t \in [a, b]$ such that $s < c < t$ with either (1) $st \in \Gamma$, or (2) $ct \in \Gamma$ and $sc \in \Gamma$. This notion of connectedness is less restrictive than that used in ref. 1, and is better suited for dealing with the interaction between ribs. The set of all graphs on $[a, b]$ is denoted $\mathscr{B}[a, b]$, and the set of all connected graphs $\mathscr{G}[a, b]$. A *lace* is a minimally connected graph, i.e., a connected graph for which the removal of any edge would result in a disconnected graph. The set of laces on $[a, b]$ is denoted by $\mathscr{L}[a, b]$. Given a connected graph Γ, the following prescription associates with Γ a unique lace $\mathscr{L}_\Gamma$: The lace $\mathscr{L}_\Gamma$ consists of bonds $s_1 t_1$, $s_2 t_2$,..., where

$$s_1 = a, \qquad t_1 = \max\{t: at \in \Gamma\}$$
$$t_{i+1} = \max\{t: st \in \Gamma, s \leqslant t_i\}$$
$$s_i = \min\{s: st_i \in \Gamma\}$$

Given a lace L, the set of all bonds $st \notin L$ such that $\mathscr{L}_{L \cup \{st\}} = L$ is denoted $\mathscr{C}(L)$. Bonds in $\mathscr{C}(L)$ are said to be *compatible* with L.

Let

$$K[a, b] = \prod_{a \leqslant s < t \leqslant b} (1 + \mathscr{U}_{st}) \tag{2.3}$$

By expanding the product in (2.3), we obtain

$$K[0, b] = \sum_{\Gamma \in \mathscr{B}[0, b]} \prod_{st \in \Gamma} \mathscr{U}_{st}$$

The contribution to the sum on the right side due to all graphs Γ for which 0 is not in an edge is exactly $K[1, b]$. To resum the contribution due to the remaining graphs, we proceed as follows. If Γ does contain an edge ending at 0, let $a(\Gamma)$ be the largest value of a such that the set of edges in Γ with an end in the interval $[0, a]$ forms a connected graph on $[0, a]$. Then resummation over graphs on $[a+1, b]$ gives

$$K[0, b] = K[1, b] + \sum_{a=1}^{b} \sum_{\Gamma \in \mathscr{G}[0, a]} \prod_{st \in \Gamma} \mathscr{U}_{st} K[a+1, b] \tag{2.4}$$

The sum over connected graphs can also be resummed:

$$\begin{aligned} \sum_{\Gamma \in \mathscr{G}[0, a]} \prod_{st \in \Gamma} \mathscr{U}_{st} &= \sum_{L \in \mathscr{L}[0, a]} \sum_{\Gamma: \mathscr{L}_\Gamma = L} \prod_{st \in L} \mathscr{U}_{st} \prod_{s't' \in \Gamma \setminus L} \mathscr{U}_{s't'} \\ &= \sum_{L \in \mathscr{L}[0, a]} \prod_{st \in L} \mathscr{U}_{st} \prod_{s't' \in \mathscr{C}(L)} (1 + \mathscr{U}_{s't'}) \equiv J[0, a] \end{aligned} \tag{2.5}$$

where the last equivalence defines $J[0, a]$. Inserting (2.5) into (2.4) yields

$$K[0, b] = K[1, b] + \sum_{a=1}^{b} J[0, a]\, K[a+1, b] \tag{2.6}$$

In (2.6) the factors $K[b, b]$ and $K[b+1, b]$ are to be interpreted as equal to 1. Separating out the $a = b$ term in (2.6) gives (for $b \geqslant 1$)

$$K[0, b] = K[1, b] + \sum_{a=1}^{b-1} J[0, a]\, K[a+1, b] + J[0, b] \tag{2.7}$$

(The middle term on the right side is taken to be 0 if $b = 1$.)

Substitution of (2.7) into (2.2) results in

$$\begin{aligned} G_z(0, x) = {} & \sum_{R_0 \ni 0} z^{|R_0|} \delta_{0,x} + \sum_{\substack{\omega: 0 \to x \\ |\omega| \geqslant 1}} z^{|\omega|} \left[\prod_{i=0}^{|\omega|} \sum_{R_i \ni \omega(i)} z^{|R_i|} \right] K[1, |\omega|] \\ & + \sum_{\substack{\omega: 0 \to x \\ |\omega| \geqslant 2}} z^{|\omega|} \left[\prod_{i=0}^{|\omega|} \sum_{R_i \ni \omega(i)} z^{|R_i|} \right] \sum_{a=1}^{|\omega|-1} J[0, a]\, K[a+1, |\omega|] \\ & + \sum_{\substack{\omega: 0 \to x \\ |\omega| \geqslant 1}} z^{|\omega|} \left[\prod_{i=0}^{|\omega|} \sum_{R_i \ni \omega(i)} z^{|R_i|} \right] J[0, |\omega|] \end{aligned} \tag{2.8}$$

The first term on the right-hand side is due to the contribution to (2.2) from the trivial zero-step walk, and the other terms are due to the walks ω with $|\omega| \geqslant 1$.

Denoting $G_z(0, 0)$ by

$$g_z \equiv G_z(0, 0) = \sum_{T \ni 0} z^{|T|} \tag{2.9}$$

and writing

$$\Pi_z(0, x) \equiv \sum_{\substack{\omega: 0 \to x \\ |\omega| \geqslant 1}} z^{|\omega|} \left[\prod_{i=0}^{|\omega|} \sum_{R_i \ni \omega(i)} z^{|R_i|} \right] J[0, |\omega|] \tag{2.10}$$

we can write the first and last terms on the right side of (2.8) as $g_z \delta_{0,x}$ and $\Pi_z(0, x)$, respectively. The second term on the right side of (2.8) is equal to

$$\sum_{R_0 \ni 0} z^{|R_0|} z \sum_{(0,u)} G_z(u, x)$$

where in the sum over $(0, u)$ we sum over nearest neighbors u of the origin for the nearest-neighbor model, and for the spread-out model over all u for which $\{0, u\}$ is a possible bond in a tree (i.e., $u \in \Omega_L \backslash \{0\}$). For the third term on the right side of (2.8), we consider ω to be composed of an initial a-step walk ω_1 from 0 to (say) u, followed by a single step to (say) v, and then a final portion (possibly consisting of 0 steps) ω_2 from v to x. The term in question is then equal to

$$z \sum_{(u,v)} \left\{ \sum_{\substack{\omega_1 : 0 \to u \\ |\omega_1| \geq 1}} z^{|\omega_1|} \left[\prod_{i=0}^{|\omega_1|} \sum_{R_i \ni \omega(i)} z^{|R_i|} \right] J[0, |\omega_1|] \right\}$$

$$\times \left\{ \sum_{\substack{\omega_2 : v \to x \\ |\omega_2| \geq 0}} z^{|\omega_2|} \left[\prod_{i=0}^{|\omega_2|} \sum_{R_i \ni \omega(i)} z^{|R_i|} \right] K[0, |\omega_2|] \right\}$$

$$= z \sum_{(u,v)} \Pi_z(0, u) \, G_z(v, x)$$

Summarizing, (2.8) can be rewritten as

$$G_z(0, x) = \delta_{0,x} g_z + \Pi_z(0, x) + z g_z \sum_{(0,u)} G_z(u, x)$$

$$+ z \sum_{(u,v)} \Pi_z(0, u) \, G_z(v, x) \tag{2.11}$$

2.1.2. The Expansion for Animals. For lattice animals the derivation of the expansion requires some modification due to the fact that an animal, unlike a tree, does not in general have a unique backbone in the sense of Section 2.1.1. The modification involves representing an animal as a "string of sausages," as has been done for percolation clusters, for example, in refs. 6 and 26. To describe this representation, some definitions are needed.

A lattice animal A containing x and y is said to have a *double connection* from x to y if there are two *distinct* (i.e., sharing no common bond) self-avoiding walks in A between x and y, or if $x = y$. A bond $\{u, v\}$ in A is called *pivotal* for the connection from x to y if its removal would disconnect the animal into two connected components with x in one connected component and y in the other. There is a natural order to the set of pivotal bonds for the connection from x to y, and each pivotal bond is ordered in a natural way, as follows. The *first* pivotal bond for the connection from x to y is the pivotal bond for which there is a double connection between one endpoint of the pivotal bond and x. The endpoint for which there is a double connection to x is then the *first* endpoint of the first pivotal bond. To determine the second pivotal bond, the role of x is then played by the second endpoint of the first pivotal bond, and so on.

Given two sites x, y and an animal A containing x and y, the *backbone* of A is now defined to be the set of pivotal bonds for the connection from x to y. In general this backbone is not connected. The *ribs* of A are the connected components which remain after the removal of the backbone from A. An example is depicted in Fig. 2. The set of all animals having a double connection between x and y is denoted $\mathscr{D}_{x,y}$. We write

$$g_z^a(x, y) = \sum_{D \in \mathscr{D}_{x,y}} z^{|D|}, \qquad g_z^a = g_z^a(0, 0) = \sum_{A \ni 0} z^{|A|} \tag{2.12}$$

Let B be an arbitrary finite ordered set of ordered bonds: $B = ((u_1, v_1),..., (u_{|B|}, v_{|B|}))$. Let $v_0 = 0$ and $u_{|B|+1} = x$. Then

$$G_z^a(0, x) = \sum_{B: |B| \geqslant 0} z^{|B|} \left[\prod_{i=0}^{|B|} \sum_{D_i \in \mathscr{D}_{v_i, u_{i+1}}} z^{|D_i|} \right] K[0, |B|]$$

where now in the definition of $K[0, |B|]$ in Eq. (2.3),

$$\mathscr{U}_{st} = \begin{cases} -1 & \text{if } D_s \text{ and } D_t \text{ share a common site} \\ 0 & \text{if } D_s \text{ and } D_t \text{ have no common site} \end{cases}$$

Define

$$\tilde{\Pi}_z^a(0, y) = \sum_{B: |B| \geqslant 1} z^{|B|} \left[\prod_{i=0}^{|B|} \sum_{D_i \in \mathscr{D}_{v_i, u_{i+1}}} z^{|D_i|} \right] J[0, |B|] \tag{2.13}$$

A calculation similar to that used to derive (2.11), using (2.6), gives

$$G_z^a(0, x) = g_z^a(0, x) + \tilde{\Pi}_z^a(0, x) + z \sum_{(u,v)} g_z^a(0, u)\, G_z^a(v, x) + z \sum_{(u,v)} \tilde{\Pi}_z^a(0, u)\, G_z^a(v, x) \tag{2.14}$$

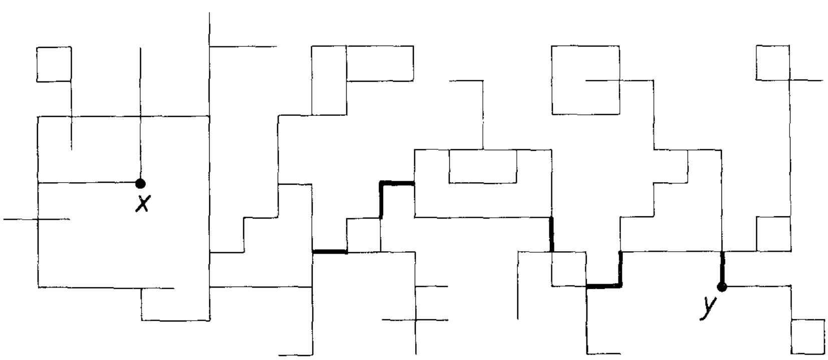


Fig. 2. Decomposition of a lattice animal A containing x and y into backbone and ribs. The backbone is drawn in bold lines.

The analogy between (2.14) and (2.11) becomes more apparent if we define

$$\Pi_z^{a,(0)}(0, u) = g_z^a(0, u)(1 - \delta_{u,0}) \tag{2.15}$$

and

$$\Pi_z^a(0, u) = \Pi_z^{a,(0)}(0, u) + \tilde{\Pi}_z^a(0, u) \tag{2.16}$$

Then (2.14) can be written

$$G_z^a(0, x) = g_z^a \delta_{0,x} + \Pi_z^a(0, x) + z g_z^a \sum_{(0,v)} G_z^a(v, x)$$
$$+ z \sum_{(u,v)} \Pi_z^a(0, u)\, G_z^a(v, x) \tag{2.17}$$

2.2. Bounds on $\Pi_z^{(a)}(0, x)$

In this section bounds are obtained for $\Pi_z^{(a)}(0, x)$ simultaneously for both the nearest-neighbor and spread-out models. The label L for spread-out models will be omitted. The bounds for lattice trees are discussed in detail in Section 2.2.1, and we comment in Section 2.2.2 on the modifications needed for lattice animals.

2.2.1. Diagrammatic Bounds on $\Pi_z(0, x)$ for Trees. We denote by $\mathscr{L}_N[0, a]$ the set of laces in $\mathscr{L}[0, a]$ consisting of exactly N edges, and write

$$J_N[0, a] = \sum_{L \in \mathscr{L}_N[0,a]} \prod_{st \in L} \mathscr{U}_{st} \prod_{s't' \in \mathscr{C}(L)} (1 + \mathscr{U}_{s't'}) \tag{2.18}$$

and

$$\Pi_z^{(N)}(0, x) = \sum_{\substack{\omega: 0 \to x \\ |\omega| \geq 1}} z^{|\omega|} \left[\prod_{i=0}^{|\omega|} \sum_{R_i \ni \omega(i)} z^{|R_i|} \right] J_N[0, |\omega|] \tag{2.19}$$

Then from (2.10) and (2.5) we have

$$\Pi_z(0, x) = \sum_{N=1}^{\infty} \Pi_z^{(N)}(0, x) \tag{2.20}$$

This is an alternating series, since for $L \in \mathscr{L}_N$, $\prod_{st \in L} \mathscr{U}_{st}$ is either $(-1)^N$ or 0. However, we will simply bound the series absolutely. For a nonzero contribution to $\Pi_z^{(N)}(0, x)$, the factor $\prod_{st \in L} \mathscr{U}_{st}$ enforces intersections between the ribs R_s and R_t, and as N increases, the number of intersections increases.

To bound the term $\Pi_z^{(1)}(0, x)$, we proceed as follows. There is a unique lace consisting of a single edge, so by definition

$$\Pi_z^{(1)}(0, x) = \sum_{\substack{\omega: 0 \to x \\ |\omega| \geqslant 1}} z^{|\omega|} \left[\prod_{i=0}^{|\omega|} \sum_{R_i \ni \omega(i)} z^{|R_i|} \right] \mathscr{U}_{0, |\omega|} \prod_{\substack{0 \leqslant s < t \leqslant |\omega| \\ (s,t) \neq (0, |\omega|)}} (1 + \mathscr{U}_{st}) \tag{2.21}$$

The factor $\mathscr{U}_{0,|\omega|}$ gives a nonzero contribution only if R_0 and $R_{|\omega|}$ intersect, and the final product in (2.21) disallows any further rib intersections. We first consider the case $x \neq 0$. Relaxing the latter restriction somewhat and overcounting an enforcement of the former gives the upper bound

$$\begin{aligned} |\Pi_z^{(1)}(0, x)| \leqslant \sum_v \sum_{\substack{\omega: 0 \to x \\ |\omega| \geqslant 1}} z^{|\omega|} \sum_{R_0 \ni 0, v} z^{|R_0|} \sum_{R_{|\omega|} \ni x, v} z^{|R_{|\omega|}|} \\ \times \left[\prod_{i=1}^{|\omega|-1} \sum_{R_i \ni \omega(i), \not\ni 0, x} z^{|R_i|} \right] \prod_{1 \leqslant s < t \leqslant |\omega| - 1} (1 + \mathscr{U}_{st}) \end{aligned} \tag{2.22}$$

Now

$$\sum_{R_0 \ni 0, v} z^{|R_0|} = G(0, v)$$

and similarly

$$\sum_{R_{|\omega|} \ni x, v} z^{|R_{|\omega|}|} = G(x, v)$$

Also,

$$\begin{aligned} &\sum_{\substack{\omega: 0 \to x \\ |\omega| \geqslant 1}} z^{|\omega|} \left[\prod_{i=1}^{|\omega|-1} \sum_{R_i \ni \omega(i), \not\ni 0, x} z^{|R_i|} \right] \prod_{1 \leqslant s < t \leqslant |\omega| - 1} (1 + \mathscr{U}_{st}) \\ &= \sum_{\substack{\omega: 0 \to x \\ |\omega| \geqslant 1}} z^{|\omega|} \left[\prod_{i=0}^{|\omega|} \sum_{R_i \ni \omega(i)} z^{|R_i|} \right] \prod_{0 \leqslant s < t \leqslant |\omega|} (1 + \mathscr{U}_{st}) \, I[|R_0| = |R_{|\omega|}| = 0] \\ &\leqslant G_z(0, x) \end{aligned}$$

Thus, for $x \neq 0$, we have

$$|\Pi_z^{(1)}(0, x)| \leqslant \sum_v G_z(0, x) \, G_z(x, v) \, G_z(v, 0) \tag{2.23}$$

When $x = 0$ in (2.21), we can find $v \in \omega$, $v \neq 0$, since $|\omega| \geqslant 1$. We divide ω into two parts ω_1 and ω_2, with $\omega_1 = (0 = \omega_1(0), \omega_1(1), \ldots, \omega_1(|\omega_1|) = v)$

and $\omega_2 = (v = \omega_2(|\omega_1|), \omega_2(|\omega_1|+1), ..., \omega_2(|\omega_1|+|\omega_2|) = 0)$. A severe overcounting with respect to v gives

$$|\Pi_z^{(1)}(0,0)| \leqslant \sum_{v \neq 0} \sum_{\substack{\omega_1: 0 \to v \\ \omega_2: v \to 0}} z^{|\omega_1|+|\omega_2|} \prod_{i=0}^{|\omega_1|} \sum_{R_i \ni \omega_1(i)} z^{|R_i|} \prod_{j=|\omega_1|+1}^{|\omega_1|+|\omega_2|} \sum_{R_j \ni \omega_2(j)} z^{|R_j|}$$

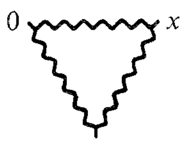
$$\times \prod_{\substack{0 \leqslant s < t \leqslant |\omega_1|+|\omega_2| \\ (s,t) \neq (0, |\omega_1|+|\omega_2|)}} (1 + \mathscr{U}_{st})$$

$$\leqslant \sum_{v \neq 0} G_z(0,v)\, G_z(v,0) \tag{2.24}$$

where in the second step we relaxed the nonintersection restriction between ribs of ω_1 and ω_2, and bounded the sums over ω_1 and ω_2 by G_z. Since $G_z(0,0) \geqslant 1$, we can now write (2.23) and (2.24) together as

$$|\Pi_z^{(1)}(0,x)| \leqslant \sum_v \{G_z(0,v)\, G_z(v,x)\, G_z(x,0) - I[0 = v = x]\, g_z^3\}$$

$$= \tag{2.25}$$



A similar strategy can be used to bound $\Pi_z^{(N)}(0,x)$ for $N > 1$. Each of the N factors in the product $\prod_{st \in L} \mathscr{U}_{st}$ in J_N imposes an intersection of ribs. The situation for $N = 2$ is shown in Fig. 3a. For a lace $L = (0t_1, s_2\,|\omega|)$ consisting of exactly two edges, there are two generic configurations possible with $\prod_{st \in L} \mathscr{U}_{st} \neq 0$, one for the case $s_2 < t_1$ and the other for $s_2 = t_1$. For general $N \geqslant 2$ there are similarly 2^{N-1} generic configurations which contribute to $\Pi_z^{(N)}$. The contribution due to each type of configuration can be bounded in essentially the same way, and we illustrate this bound in detail only for the case of $N = 2$ with $s_2 = t_1$.

The contribution to $\Pi_z^{(2)}$ due to laces with $s_2 = t_1$ can be written

$$\sum_{\substack{\omega: 0 \to x \\ |\omega| \geqslant 2}} z^{|\omega|} \left[\prod_{i=0}^{|\omega|} \sum_{R_i \ni \omega(i)} z^{|R_i|} \right] \sum_{s=1}^{|\omega|-1} \mathscr{U}_{0s} \mathscr{U}_{s|\omega|} \prod_{s't' \in \mathscr{C}(0s, s|\omega|)} (1 + \mathscr{U}_{s't'})$$

$$\leqslant \sum_{\substack{\omega: 0 \to x \\ |\omega| \geqslant 2}} z^{|\omega|} \left[\prod_{i=0}^{|\omega|} \sum_{R_i \ni \omega(i)} z^{|R_i|} \right] \sum_{s=1}^{|\omega|-1} \mathscr{U}_{0s} \mathscr{U}_{s|\omega|} K[0,s]\, K[s, |\omega|] \tag{2.26}$$

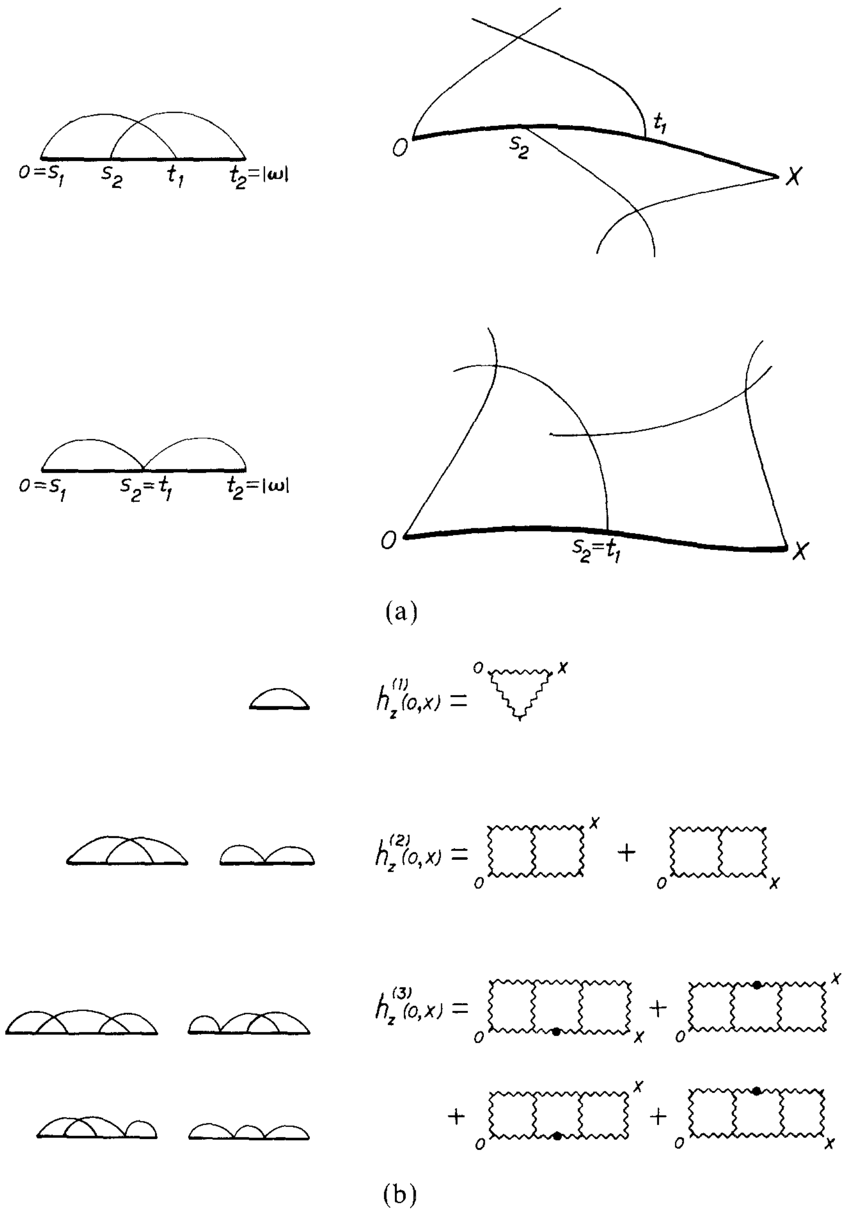


Fig. 3. (a) The two generic laces consisting of two bonds, and a schematic diagram showing the corresponding rib intersections for a nonzero contribution to $\Pi_z^{(2)}(0, x)$. (b) Diagrammatic representation for $h_z^{(N)}$, $N=1, 2, 3$. Sums over vertices are constrained to disallow the coincidence of all vertices on any loop. On the left, the laces giving rise to the corresponding terms in $h_z^{(N)}$ are shown.

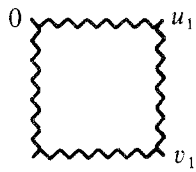
Let y be a site where R_0 and R_s intersect, and let $\beta_{R_s}(\omega(s), y)$ be the backbone of R_s. For $\mathscr{U}_{s|\omega|} \neq 0$, $R_{|\omega|}$ must intersect R_s, and hence there must be a rib emanating from a site on the backbone $\beta_{R_s}(\omega(s), y)$ which intersects $R_{|\omega|}$ (see Fig. 3a). Now, by arguing in a similar fashion to the case $N = 1$, (2.26) can be bounded above by the second term in $h_z^{(2)}(0, x)$, depicted in Fig. 3b.

In general

$$|\Pi_z^{(N)}(0, x)| \leqslant h_z^{(N)}(0, x) \tag{2.27}$$

where for $N \geqslant 1$, $h_z^{(N)}(0, x)$ is given by a sum of 2^{N-1} diagrams, each containing exactly N nontrivial loops. These diagrams are shown in Fig. 3b for $N = 1, 2, 3$. The propagators in the diagrams are all *independent*. More precisely, for $u_i, v_i \in \mathbf{Z}^d$, let

$A_1(0, u_1, v_1) \equiv$

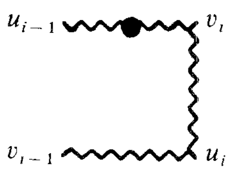


$$= \sum_y (G_z(0, u_1)\, G_z(u_1, v_1)\, G_z(v_1, y)\, G_z(y, 0) - I[0 = u_1 = v_1 = y]\, g_z^4)$$

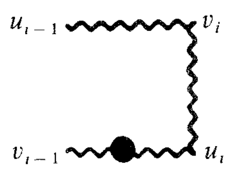
$A_2(u_{i-1}, v_{i-1}, u_i, v_i)$

$$\equiv A_2^{(1)}(u_{i-1}, v_{i-1}, u_i, v_i) + A_2^{(2)}(u_{i-1}, v_{i-1}, u_i, v_i)$$

$\equiv$



$+$



$$= \sum_y (G_z(u_{i-1}, y)\, G_z(y, v_i)\, G_z(v_i, u_i)\, G_z(u_i, v_{i-1})$$

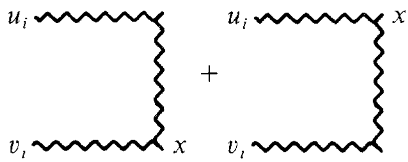
$$- I[v_{i-1} = u_{i-1} = v_i = u_i = y]\, g_z^4)$$

$$+ \sum_y (G_z(u_{i-1}, v_i)\, G_z(v_i, u_i)\, G_z(u_i, y)\, G_z(y, v_{i-1})$$

$$- I[v_{i-1} = u_{i-1} = v_i = u_i = y]\, g_z^4)$$

and

$$A_3(v_i, u_i, x) \equiv A_3^{(1)}(v_i, u_i, x) + A_3^{(2)}(v_i, u_i, x)$$



$$= \sum_y [G_z(u_i, y)\, G_z(y, x)\, G_z(x, v_i) - I[v_i = x = u_i = y]\, g_z^3]$$

$$+ \sum_y [G_z(u_i, x)\, G_z(x, y)\, G_z(y, v_i) - I[v_i = x = u_i = y]\, g_z^3]$$

Then

$$h_z^{(1)}(0, x) = \tfrac{1}{2} A_3(0, 0, x) \tag{2.28}$$

and for $N \geqslant 2$,

$$h_z^{(N)}(0, x) = \sum_{u_1, v_1, \ldots, u_{N-1}, v_{N-1}} A_1(0, u_1, v_1) \times \prod_{j=2}^{N-1} A_2(u_{j-1}, v_{j-1}, u_j, v_j)\, A_3(v_{N-1}, u_{N-1}, x) \tag{2.29}$$

Combining (2.27) and (2.20) gives the fundamental bound

$$|\Pi_z(0, x)| \leqslant \sum_{N=1}^{\infty} h_z^{(N)}(0, x) \tag{2.30}$$

2.2.2. Diagrammatic Bounds on $\Pi_z^a(0, x)$ for Animals. For lattice animals the diagrammatic bounds on Π_z^a are more complex than the corresponding bound (2.30) for trees, due to the more intricate nature of possible rib–rib intersections for animals. The diagrams which arise in bounding Π_z^a are closely related to the diagrams encountered in ref. 6 for the analogous percolation problem.

As in (2.19), we define $\Pi_z^{a,(N)}$ by replacing J by J_N in (2.13), for $N \geqslant 1$. In addition, there is the term $\Pi_z^{a,(0)}$ from (2.16). Thus, we have

$$\Pi_z^a(0, x) = \sum_{N=0}^{\infty} \Pi_z^{a,(N)}(0, x)$$

The basic tool used to estimate $\Pi_z^{a,(N)}$ is the lattice animal version of the van den Berg–Kesten inequality[27] stated in the following lemma. The proof of the lemma is essentially the same as the proof of the first-order skeleton inequality sketched in Section 1.3.

Lemma 2.1. Given sets of lattice paths $E_1,..., E_n$, let $\mathscr{A}_i$ $(i = 1,..., n)$ be the set of all animals which contain a path in E_i, and let $\mathscr{A}$ be the set of all animals which contain distinct (i.e., sharing no common bond) paths in each of $E_1,..., E_n$. Then

$$\sum_{A \in \mathscr{A}} z^{|A|} \leqslant \prod_{i=1}^{n} \left[\sum_{A_i \in \mathscr{A}_i} z^{|A_i|} \right]$$

Proof. Given any animal $A \in \mathscr{A}$, it is possible to decompose A into animals $A_1,..., A_n$ which are distinct in the sense that no two share a common bond, and with $A_i \in \mathscr{A}_i$ $(i = 1,..., n)$. This decomposition is in general not unique, but it can be made unique if the decomposition is formed via some specific algorithm. An example of such an algorithm is the following: (i) Out of all possible decompositions, consider those for which $|A_1|$ is largest, and of these, consider those for which $|A_2|$ is largest, and so on. (ii) From the reduced set of decompositions obtained in step (i), choose the A_1 which is lexicographically largest in the sense that it contains the lexicographically largest bond not found in any other A_1; then repeat for the compatible A_2's, and so on.

This provides a mapping which associates to each animal $A \in \mathscr{A}$ an n-tuple of animals $A_1,..., A_n$ with $|A_1| + |A_2| + \cdots + |A_n| = |A|$ and $A_i \in \mathscr{A}_i$. Since $A = A_1 \cup \cdots \cup A_n$, each resulting n-tuple corresponds to exactly one animal $A \in \mathscr{A}$. Hence, by overcounting we have

$$\sum_{A \in \mathscr{A}} z^{|A|} \leqslant \sum_{A_1 \in \mathscr{A}_1} \cdots \sum_{A_n \in \mathscr{A}_n} z^{|A_1| + |A_2| + \cdots + |A_n|} \qquad \blacksquare$$

Now, to bound the $N = 0$ term $\Pi_z^{a,(0)}(0, x)$, we simply note that by Lemma 2.1 and the definition of $g_z^a(0, x)$ in (2.12),

$$\begin{aligned} \Pi_z^{a,(0)}(0, x) &= \sum_{D \in \mathscr{D}_{0,x}} z^{|D|}[1 - \delta_{0,x}] \\ &\leqslant G_z^a(0, x)^2 (1 - \delta_{0,x}) \equiv h_z^{a,(0)}(0, x) \end{aligned} \tag{2.31}$$

For the $N = 1$ term, by definition we have

$$\Pi_z^{a,(1)}(0, x) = \sum_{B:|B| \geqslant 1} z^{|B|} \left[\prod_{i=0}^{|B|} \sum_{D_i \in \mathscr{D}_{v_i, u_{i+1}}} z^{|D_i|} \right] \mathscr{U}_{0|B|} \prod_{\substack{0 \leqslant s < t \leqslant |B| \\ st \neq 0|B|}} (1 + \mathscr{U}_{st})$$

The final product on the right side is bounded above by

$$\prod_{1 \leqslant s < t \leqslant |B|-1} (1 + \mathscr{U}_{st})\, I[D_i \not\ni u_1, v_{|B|};\, i = 1,..., |B|-1]$$

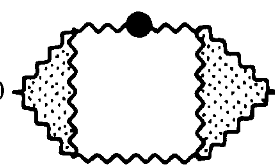
and hence

$$\begin{aligned} |\Pi_z^{a,(1)}(0, x)| &\leqslant \sum_{u_1, v_{|B|}} G_z^a(u_1, v_{|B|}) \sum_{D_0 \in \mathscr{D}_{0,u_1}} z^{|D_0|} \\ &\quad \times \sum_{D_{|B|} \in \mathscr{D}_{v_{|B|},x}} z^{|D_{|B|}|} I[D_0 \cap D_{|B|} \neq \varnothing] \\ &\leqslant \sum_{u_1, v_{|B|}} G_z^a(u_1, v_{|B|}) \sum_{y} \sum_{D_0 \in \mathscr{D}_{0,u_1}} z^{|D_0|} I[D_0 \ni y] \\ &\quad \times \sum_{D_{|B|} \in \mathscr{D}_{v_{|B|},x}} z^{|D_{|B|}|} I[D_{|B|} \ni y] \end{aligned} \tag{2.32}$$

For $D_0 \in \mathscr{D}_{0,u_1}$ to contain the site y, there must be a site w_1 and distinct paths in D_0 from 0 to u_1, 0 to w_1, w_1 to y, and w_1 to u_1. Thus, by Lemma 2.1,

$$\sum_{D_0 \in \mathscr{D}_{0,u_1}} z^{|D_0|} I[D_0 \ni y] \leqslant G_z^a(0, u_1) \sum_{w_1} G_z^a(0, w_1)\, G_z^a(w_1, u_1)\, G_z^a(w_1, y)$$

Employing the same analysis on the other factor of (2.32) gives

$\Pi_z^{a,(1)}(0, x) \leqslant 0$  $x \equiv h_z^{a,(1)}(0, x)$

Similarly, it can be shown that

$$\Pi_z^{a,(N)}(0, x) \leqslant h_z^{a,(N)}(0, x), \qquad N \geqslant 0 \tag{2.33}$$

where, for $N \geqslant 2$, $h_z^{a,(N)}(0, x)$ is given by a sum of 5^{N-1} diagrams, which are related to the corresponding diagrams for percolation. This leads to

$$|\Pi_z^a(0, x)| \leqslant \sum_{N=0}^{\infty} h_z^{a,(N)}(0, x) \tag{2.34}$$

We illustrate the diagrams contributing to $h_z^{a,(2)}$ in Fig. 4, but omit the detailed description of $h_z^{a,(N)}$ for $N > 2$. While $h_z^{a,(1)}$ can be bounded using only the triangle diagram, higher orders require the square diagram.

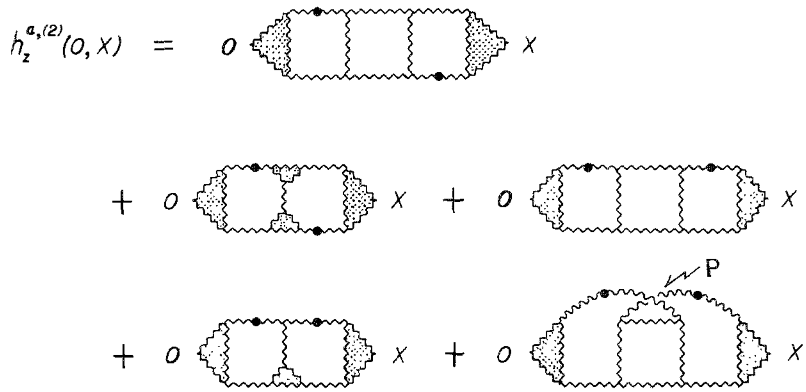


Fig. 4. Diagrammatic representation of $h_z^{a,(2)}(0,x)$. Sums over vertices on unshaded loops are constrained to disallow the coincidence of all vertices on the loop. In the last diagram, there is no vertex at P.

2.3. Bounds on $h_z^{(N)}(0, x)$

In this section, bounds are obtained for $h_z^{(N)}(0, x)$ for both the nearest-neighbor and spread-out models of trees. As usual, the label L for spread-out models is omitted. Bounds on $h_z^{a,(N)}(0, x)$ can be obtained in a similar fashion, but we omit the details of these lattice animal bounds.

We need bounds on $\sum_x h_z^{(N)}(0, x)$ and $\sum_x |x|^2 h_z^{(N)}(0, x)$. The first of these will be used with (2.30) to bound $\Pi_z(0, x)$, and the second will be used to bound the second derivative with respect to k_μ of the Fourier transform

$$\hat{\Pi}_z(k) \equiv \sum_x \Pi_z(0, x)\, e^{ik \cdot x}$$

The basic operation used in obtaining the bounds is to repeatedly apply the simple inequality

$$\left|\sum_x f(x)\, g(x)\right| \leqslant \sup_x |f(x)| \sum_y |g(y)| \tag{2.35}$$

The method closely mirrors that used in refs. 1 and 6 to estimate analogous diagrams which occur for self-avoiding walk and percolation, respectively.

The upper bounds obtained are in terms of the quantities given in the following definition.

Definition 2.2. For $l \in \mathbf{Z}^d$ we define

$$T_l = \sum_{x,y} G_z(0,x)\, G_z(x,y)\, G_z(y,l) - \delta_{0,l}(g_z)^3$$

$$W_l = \sum_{x,y} |x|^2\, G_z(0,x)\, G_z(x,y)\, G_z(y,l)$$

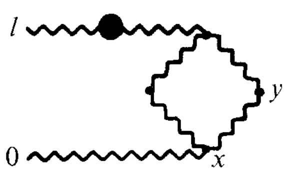
$$S_l = \sum_{w,x,y} G_z(0,w)\, G_z(w,x)\, G_z(x,y)\, G_z(y,l) - \delta_{0,l}(g_z)^4$$

$$S = S_0, \qquad T = T_0, \qquad W = W_0$$

$$\bar{S} = \sup_{l \in \mathbf{Z}^d} S_l, \qquad \bar{T} = \sup_{l \in \mathbf{Z}^d} T_l, \qquad \bar{W} = \sup_{l \in \mathbf{Z}^d} W_l$$

The following lemma will be used with (2.30) to bound $\hat{\Pi}_z(k)$ and $\partial_\mu^s \hat{\Pi}_z(k)$. A similar lemma can be proved for lattice animals, involving more complicated upper bounds analogous to those encountered in percolation.[6] The major difference for lattice animals is that the upper bound corresponding to (2.38) involves an additional quantity

$$\bar{H} \equiv \sup_{l \in \mathbf{Z}^d} H_l \equiv \sup_{l \in \mathbf{Z}^d} \sum_{x,y} |x-y|^2 \qquad (2.36)$$



which is needed to bound diagrams like the last diagram in Fig. 4.

Lemma 2.3. (a)

$$\sum_x h_z^{(N)}(0,x) \leqslant \begin{cases} T, & N=1 \\ 2^{N-1}\bar{T}S\bar{S}^{N-2}, & N \geqslant 2 \end{cases} \qquad (2.37)$$

(b)

$$\sum_x |x|^2\, h_z^{(N)}(0,x) \leqslant \begin{cases} W, & N=1 \\ 2^{N+1}N^2\bar{W}S\bar{S}^{N-2}, & N \geqslant 2 \end{cases} \qquad (2.38)$$

Proof. (a) The case $N=1$ follows immediately from (2.28). For $N \geqslant 2$ we sum (2.29) over x and apply (2.35) repeatedly to obtain

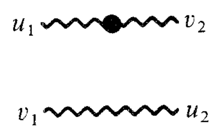
$$\begin{aligned} \sum_x h_z^{(N)}(0,x) &\leqslant \sum_{u_1,v_1} A_1(0,u_1,v_1) \cdot \prod_{j=2}^{N-1} \left[\sup_{w_{j-1}} \sum_{u_j,v_j} A_2(0,w_{j-1},u_j,v_j) \right] \\ &\quad \times \sup_{w_{N-1}} \sum_x A_3(0,w_{N-1},x) \\ &\leqslant S(2\bar{S})^{N-2}\, 2\bar{T} \end{aligned}$$

(b) For $N=1$ the desired inequality follows from (2.28). To illustrate the method for $N \geqslant 2$, we consider in detail only the case $N=3$; other values of N are handled similarly.

For $N=3$, by (2.29) and the triangle inequality we have

$$\sum_x |x|^2 h_z^{(3)}(0, x)$$
$$\leqslant 3 \sum_x \sum_{u_1, v_1, u_2, v_2} (|u_1|^2 + |v_2 - u_1|^2 + |x - v_2|^2)$$
$$\times A_1(0, u_1, v_1) A_2(u_1, v_1, u_2, v_2) A_3(v_2, u_2, x) \qquad (2.39)$$

Substitution of $A_2^{(1)} + A_2^{(2)}$ for A_2 and $A_3^{(1)} + A_3^{(2)}$ for A_3 gives four terms, of which we consider only the term T_1 involving $A_2^{(1)}$ and $A_3^{(1)}$. A similar argument gives the same upper bound as that we will obtain for T_1 for the other three terms. Define

$B(u_1, v_1, u_2, v_2) =$ 

$$= G_z(v_1, u_2) \sum_w G_z(u_1, w)\, G_z(w, v_2)$$

A regrouping of the propagators contributing to T_1 gives

$$T_1 = 3 \sum_x \sum_{u_1, v_1, u_2, v_2} \{ |u_1|^2 A_3^{(1)}(u_1, v_1, 0)\, A_2^{(1)}(v_2, u_2, v_1, u_1)\, A_1(x, v_2, u_2)$$
$$+ A_1(0, u_1, v_1)\, |v_2 - u_1|^2\, B(u_1, v_1, u_2, v_2)\, A_1(x, v_2, u_2)$$
$$+ A_1(0, u_1, v_1)\, A_2^{(1)}(u_1, v_1, u_2, v_2)\, |x - v_2|^2\, A_3^{(1)}(v_2, u_2, x)\} \qquad (2.40)$$

The first and last terms on the right side are bounded by $\bar{W}\bar{S}S$, and the second term is bounded by $4\bar{W}S^2$. We now illustrate this for the second term. The other terms are similar. First the triangle inequality is applied again in the form $|v_2 - u_1|^2 \leqslant 2(|v_2 - w|^2 + |w - u_1|^2)$. Each of the resulting terms is bounded in the same way. For example, to estimate the $|v_2 - w|^2$ term, we use translation invariance and (2.35) to obtain the bound

$$\sum_{u_1, v_1} A_1(0, u_1, v_1) \sup_a \left[\sum_{w, v_2} |v_2|^2 G_z(0, v_2)\, G_z(v_2, w)\, G_z(w, a) \right]$$
$$\times \sum_{x, u_2} A_1(0, x, u_2)$$
$$\leqslant S\bar{W}S$$

This gives $T_1 \leqslant 3(2\bar{W}\bar{S}S + 4S^2\bar{W}) \leqslant 18S\bar{S}\bar{W}$, and hence (2.39) is bounded above by $72S\bar{S}\bar{W}$.

A factor of 2^{N-1} in the bound (2.38) for $N \geqslant 2$ comes from the 2^{N-1} diagrams in $h_z^{(N)}$, and the factor of N^2 from application of the triangle inequality as in (2.39). The additional factor of 2^2 accounts for the need to apply the triangle inequality as for the second term in (2.40). ∎

Combining Lemma 2.3 with (2.30) leads to the following result.

Lemma 2.4. (a)

$$|\partial_\mu^s \hat{\Pi}_z(k)| \leqslant \begin{cases} T + 2\bar{T}S \sum_{N=2}^{\infty} (2\bar{S})^{N-2}, & s=0 \\ W + 8\bar{W}S \sum_{N=2}^{\infty} N^2(2\bar{S})^{N-2}, & s=1,2 \end{cases} \tag{2.41}$$

(b)

$$|\hat{\Pi}_z(0) - \hat{\Pi}_z(k)| \leqslant \frac{1}{2d} \sum_{\mu=1}^{d} k_\mu^2 \left[W + 8\bar{W}S \sum_{N=2}^{\infty} N^2(2\bar{S})^{N-2} \right] \tag{2.42}$$

Proof. (a) By definition, $\hat{\Pi}_z(k) = \sum_x \Pi_z(0, x)\, e^{ik \cdot x}$, and (2.41) follows immediately from (2.30) and Lemma 2.3. (For $s = 1$, we use the fact that for $x \in \mathbf{Z}^d$, $|x| \leqslant |x|^2$.)

(b) By symmetry, (2.30), and (2.38),

$$\begin{aligned} |\hat{\Pi}_z(0) - \hat{\Pi}_z(k)| &= \left| \sum_x (1 - \cos k \cdot x) \sum_{N=1}^{\infty} \Pi_z^{(N)}(0, x) \right| \\ &\leqslant \sum_{N=1}^{\infty} \sum_x \frac{1}{2} \left(\sum_{\mu=1}^{d} k_\mu x_\mu \right)^2 h_z^{(N)}(0, x) \\ &= \frac{1}{2} \sum_{N=1}^{\infty} \sum_{\mu=1}^{d} k_\mu^2 \sum_x x_\mu^2 h_z^{(N)}(0, x) \\ &\leqslant \frac{1}{2d} \sum_{\mu=1}^{d} k_\mu^2 \left[W + 8\bar{W}S \sum_{N=2}^{\infty} N^2(2\bar{S})^{N-2} \right] \end{aligned}$$ ∎

3. PROOF OF THEOREMS 1.1 AND 1.2

Since finiteness of the square diagram at the critical point is known to imply that $\gamma = 1/2$ in the sense that (1.15) holds,[16–18] to prove Theorems 1.1(a) and 1.2(a) it suffices to prove that the square diagram is finite at the critical point, and to obtain the infrared bound. By the monotone convergence theorem, a bound on the square diagram which is uniform in $z < z_c^{(a)}$ implies the same bound at the critical point [see (3.3)]. The bulk of the proof consists in obtaining such a uniform bound. In the course of the proof of this uniform bound, the infrared bound is also proven. As will be shown in Section 3.3, the proof also yields Theorems 1.1(b) and 1.2(b), with little more effort.

We give the proof of these results only for the case of spread-out lattice trees above eight dimensions. With minor changes, the same methods apply to nearest-neighbor trees and animals in sufficiently high dimensions, and to spread-out lattice animals above eight dimensions. The proof has the same general structure as that used for the self-avoiding walk in ref. 2 and for percolation in ref. 6, although the details do vary.

3.1. General Structure of the Proof

For the remainder of Section 3 we restrict our attention to spread-out lattice trees in a fixed dimension $d > 8$. A uniform bound on the square diagram below the critical point is a consequence of the following Lemma 3.1, Lemma 3.2, and Proposition 3.3. Before stating these results, we introduce some notation. Let $|\Omega_L|$ denote the cardinality of $(\mathbf{Z}^d \cap \Omega_L)\backslash\{0\}$, and let

$$z_L = \frac{(2\pi)^{1/2}}{3e\,|\Omega_L|}$$

Then $z_L \sim L^{-d}$. Define

$$\hat{D}_L(k) = \frac{1}{|\Omega_L|} \sum_{x \in (\mathbf{Z}^d \cap \Omega_L)\backslash\{0\}} e^{ik \cdot x}$$

and

$$C_L(x, y) = \sum_{\omega: x \to y} \left(\frac{1}{|\Omega_L|}\right)^{|\omega|} = \int_{[-\pi, \pi]^d} \frac{d^d k}{(2\pi)^d} \frac{e^{ik \cdot (y-x)}}{1 - \hat{D}_L(k)}$$

Here the sum over ω is the sum over all simple random walks from x to y, with $\omega(i+1) - \omega(i) \in \Omega_L\backslash\{0\}$. To simplify the notation, we will usually drop the label L. Recall that by definition $g_z = G_z(0, 0)$. The three basic results are the following.

Lemma 3.1. The two-point function satisfies the bound

$$G_z(x, y) \leqslant g_z C_L(x, y) \qquad \text{for} \quad z \leqslant z_L \tag{3.1}$$

In addition,

$$g_z \leqslant \frac{3e}{(2\pi)^{1/2}} \qquad \text{for} \quad z \leqslant \frac{1}{e\,|\Omega_L|} \tag{3.2}$$

Since g_z is a power series with positive coefficients, it follows that dg_z/dz is finite for $z < (e\,|\Omega_L|)^{-1}$. Since, by (1.1), $\chi(z) \leqslant (d/dz)[zg_z]$, it follows that $\chi(z) < \infty$ for $z < (e\,|\Omega_L|)^{-1}$, and hence $z_c \geqslant (e\,|\Omega_L|)^{-1} > z_L$.

Lemma 3.2. For $z < z_c$, S, W_l, and g_z are continuous in z.

Proposition 3.3. For any $d > 8$ there is an $L_0 = L_0(d, \Omega_1)$ such that if $L \geqslant L_0$, then for any fixed $z \in [z_L, z_c)$,

$$P_4 \Rightarrow P_3$$

where P_α is the statement that the following inequalities hold:

$$S \leqslant 400\alpha \cdot K_S \cdot L^{1-d}, \qquad W \leqslant 4000\alpha \cdot K_W \cdot L^{3-d}, \qquad g_z \leqslant \tfrac{4}{3}\alpha$$

$$W_l \leqslant \alpha K' L^{3-d} \qquad \text{for} \quad \|l\|_1 \leqslant M(z, L)$$

In Proposition 3.3, K_S and K_W are constants which are defined in the Appendix, Lemma A.1. They are defined in terms of the Gaussian analogues S_G and W_G of S and W, given by

$$S_G \equiv \sum_{x, y, w} C_L(0, x)\, C_L(x, y)\, C_L(y, w)\, C_L(w, 0) - C_L(0, 0)^4$$

and

$$W_{G,l} \equiv \sum_{x, y} |x|^2\, C_L(0, x)\, C_L(x, y)\, C_L(y, l)$$

The numerical constants appearing in P_α are somewhat arbitrary. The (large) constant K' is determined in the proof of the Proposition. The value of $M(z, L)$ is chosen sufficiently large so that $W_l \leqslant 3K'L^{3-d}$ for $\|l\|_1 > M(z, L)$. This is possible because the two-point function decays exponentially for $z < z_c$ (with a decay rate which depends on z[16]). The statement P_α is, for fixed $z < z_c$, a statement of *finitely many* inequalities.

For lattice animals (as for percolation[6]), P_α must be augmented with a bound on the quantity H_l defined in (2.36).

We now explain how together Lemmas 3.1 and 3.2 and Proposition 3.3 imply a uniform bound on the square diagram below the critical point. First, Lemma 3.1 implies that P_3 holds for $z \leqslant z_L$. To see this, note that it follows from the definition of K_W in (A.4) that $W_G \leqslant K_W L^{3-d}$, and hence by Lemma 3.1 we have $W \leqslant g_z^3 W_G \leqslant 64 K_W L^{3-d}$. The inequality $g_z \leqslant 4$ follows from (3.2). For S, it follows from (3.1), (3.2), and (A.3) that

$$S = \sum_{w,x,y} G_L(0, x)\, G_L(x, y)\, G_L(y, w)\, G_L(w, 0)\{1 - I[x = y = w = 0]\}$$

$$\leqslant (g_z)^4 \left[\sum_{w,x,y} C_L(0, x)\, C_L(x, y)\, C_L(y, w)\, C_L(w, 0) - C_L(0, 0)^4 \right]$$

$$= (g_z)^4\, S_G \leqslant \frac{81 e^4}{4\pi^2} K_S L^{-d+1}$$

which gives the desired inequality for S. For W_l, it is sufficient to show that $g_z^3 W_{G,l} \leqslant 4^3 W_{G,l}$ obeys the bound of P_3 for all $l \in \mathbf{Z}^d$. This can be done by the argument of part (d) of the proof of Proposition 3.3, in Section 3.2, by putting $|z| \cdot |\Omega_L| = 1$, $g_z = 1$, $\hat{\Pi}_z(k) = 0$ there.

Now it follows from Proposition 3.3 (together with Lemmas 3.1 and 3.2) that there are forbidden regions in the graphs of S, W, W_l, and g versus z. For example, the graph of S cannot enter the rectangle $[z_L, z_c) \times (1200 K_S L^{-d+1},\ 1600 K_S L^{-d+1})$, and hence $S \leqslant 1200 K_S L^{-d+1}$ for all $z < z_c$. More generally, P_3 holds for all $z < z_c$. It follows that

$$\square(z) = S + (g_z)^4 \leqslant 1200 K_S L^{-d+1} + 4^4 \qquad \text{for} \quad z < z_c$$

Since

$$\square(z) = \sum_{w,x,y}\ \sum_{T_1 \ni 0, w}\ \sum_{T_2 \ni w, x}\ \sum_{T_3 \ni x, y}\ \sum_{T_4 \ni y, 0} z^{|T_1| + |T_2| + |T_3| + |T_4|} \tag{3.3}$$

it follows from the monotone convergence theorem that $\square(z_c) = \lim_{z \nearrow z_c} \square(z) < \infty$. Note that we have in fact proven that $S \leqslant 1200 K_S L^{-d+1} \ll 1$, as promised in Section 1.3.

In the course of the proof it will be shown that the infrared bound follows from P_4. Since we will prove the stronger statement P_3, this proves the infrared bound. The proof that $\nu = 1/4$ will be given in Section 3.3. In the next section we prove Proposition 3.3. Lemma 3.2 is an immediate consequence of the monotone convergence theorem, together with the fact that the two-point function decays exponentially below the critical point. Lemma 3.1 is proven as follows.

Proof of Lemma 3.1. We give the proof for both spread-out trees and animals. By definition, $G_z(x, y) = \sum_{C \ni x, y} z^{|C|}$, where the sum over C denotes a sum over all trees or animals (depending on which model is being considered) containing x and y which can be constructed from bonds $\{u, v\}$ with $v-u \in \Omega_L \backslash \{0\}$. This sum is bounded above as follows:

$$G_z(x, y) \leqslant \sum_{\omega: x \to y}' z^{|\omega|}(g_z)^{|\omega|+1} \tag{3.4}$$

where the sum $\sum'$ is over all self-avoiding walks from x to y. Hence, for z such that $zg_z \leqslant 1/|\Omega_L|$, it follows from (3.4) that

$$G_z(x, y) \leqslant g_z \sum_{\omega. x \to y}' \left(\frac{1}{|\Omega_L|}\right)^{|\omega|}$$

Bounding the sum over all self-avoiding walks by the sum over all simple random walks gives (3.1) for z such that $zg_z \leqslant 1/|\Omega_L|$.

Now g_z (either for trees or animals) is bounded above by the corresponding sum over all embedded *abstract* trees, such that the origin is a vertex of the embedded tree. Thus, by Cayley's theorem (ref. 28, p. 75) and the inequality $n! \geqslant n^n e^{-n}(2\pi n)^{1/2}$,

$$g_z \leqslant \sum_{n=0}^{\infty} z^n |\Omega_L|^n (n+1) \frac{(n+1)^{n-1}}{(n+1)!} \leqslant \sum_{n=1}^{\infty} \frac{e(z|\Omega_L| e)^{n-1}}{(2\pi)^{1/2} n^{3/2}} \tag{3.5}$$

The first inequality follows from the fact that there are $(n+1)^{n-1}/(n+1)!$ unlabeled abstract trees with n edges; the factor of $(n+1)$ is due to the fact that any vertex can be mapped to the origin, and the factor $|\Omega_L|^n$ corresponds to the fact that each edge of the abstract tree can be mapped to at most $|\Omega_L|$ bonds. By (3.5),

$$g_z \leqslant \frac{e}{(2\pi)^{1/2}} \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \leqslant \frac{3e}{(2\pi)^{1/2}} \qquad \text{for} \quad z \leqslant \frac{1}{e|\Omega_L|}$$

which gives (3.2). Hence, for $z \leqslant z_L$, $zg_z \leqslant 1/|\Omega_L|$ and (3.1) holds. ∎

3.2. Proof of Proposition 3.3

In this section we prove Proposition 3.3, which together with Lemmas 3.1 and 3.2 implies that P_3 holds. We will also show that the infrared bound follows from P_4. This is sufficient to prove the infrared bound, since the stronger statement P_3 holds. As usual in Section 3, we restrict our attention to spread-out lattice trees. We begin with a lemma.

Lemma 3.4. If $z < z_c$ and we assume P_4, then there is a constant c_1 (which does not depend on K') such that

$$\bar{S} \equiv \sup_l S_l \leqslant c_1 L^{(1-d)/2}$$

Proof. By P_4, $S_0 \leqslant cL^{(1-d)/2}$ ($c = 1600K_S$), so it suffices to consider S_l for $l \neq 0$. Fix $l \neq 0$. Then

$$\begin{aligned}
S_l &= \int \frac{d^dk}{(2\pi)^d} [\hat{G}_z(k)^4 - (g_z)^4]\, e^{ik\cdot l} \\
&= \int \frac{d^dk}{(2\pi)^d} ([\hat{G}_z(k) - g_z]^2 \{[\hat{G}_z(k) + g_z]^2 + 2(g_z)^2\} \\
&\quad + 4(g_z)^3 \hat{G}_z(k) - 4(g_z)^4)\, e^{ik\cdot l} \\
&\leqslant \int \frac{d^dk}{(2\pi)^d} [\hat{G}_z(k) - g_z]^2 \{[\hat{G}_z(k) + g_z]^2 + 2(g_z)^2\} + 4(g_z)^3 G_z(0, l) \\
&= S + 4(g_z)^3 G_z(0, l)
\end{aligned}$$

(All integrals with respect to k extend over $[-\pi, \pi]^d$.) Now, by symmetry,

$$G_z(0, l)^2 \leqslant \frac{1}{2d} \sum_{x \neq 0} G_z(0, x)^2 \leqslant \frac{1}{2d} \frac{1}{6} S$$

so by P_4

$$S_l \leqslant S + 4\left(\frac{16}{3}\right)^3 \left(\frac{S}{12d}\right)^{1/2} \leqslant c_1 L^{(1-d)/2} \quad \blacksquare$$

By Lemma 3.4, it follows from (2.41), (2.42), the fact that $\bar{T} \leqslant \frac{1}{4}\bar{S}$, and the assumption P_4 that for L sufficiently large there are constants c_2, c_3, and c_4 (which do not depend on K') such that

$$|\partial_\mu^s \hat{\Pi}_z(k)| \leqslant \begin{cases} c_2 L^{1-d}, & s = 0 \\ c_3 L^{3-d}, & s = 1, 2 \end{cases} \tag{3.6}$$

and

$$|\hat{\Pi}_z(0) - \hat{\Pi}_z(k)| \leqslant c_4 L^{3-d} k^2 \tag{3.7}$$

These bounds will be used to show that, for fixed $z \in [z_L, z_c)$, P_4 implies P_3.

Fix $z \in [z_L, z_c)$. Using the fact that the Fourier transform of a convolution is the product of Fourier transforms, taking the Fourier transform of (2.11), and solving for $\hat{G}_z(k)$ gives

$$\hat{G}_z(k) = \frac{g_z + \hat{\Pi}_z(k)}{1 - z\,|\Omega_L|\, \hat{D}_L(k)[g_z + \hat{\Pi}_z(k)]} \equiv \frac{g_z + \hat{\Pi}_z(k)}{\hat{F}_z(k)} \tag{3.8}$$

This equation implicitly defines $\hat{F}_z(k)$. Since $\hat{G}_z(0) = \chi(z) \in (0, \infty)$, it follows from (3.8), (3.6), and the fact that $g_z \geqslant 1$ that for L sufficiently large

$$\hat{F}_z(0) = 1 - z\,|\Omega_L|\,[g_z + \hat{\Pi}_z(0)] > 0 \tag{3.9}$$

Therefore

$$\begin{aligned}
\hat{G}_z(k) &= \frac{g_z + \hat{\Pi}_z(k)}{\hat{F}_z(k) - \hat{F}_z(0) + \hat{F}_z(0)} \\
&\leqslant \frac{g_z + \hat{\Pi}_z(k)}{\hat{F}_z(k) - \hat{F}_z(0)} \\
&= \frac{1}{z\,|\Omega_L|} \frac{1 + g_z^{-1}\hat{\Pi}_z(k)}{[1 - \hat{D}_L(k)] + (g_z)^{-1}\,[\hat{\Pi}_z(0) - \hat{\Pi}_z(k)\,\hat{D}_L(k)]} \\
&\leqslant \frac{3e}{(2\pi)^{1/2}} \frac{1 + g_z^{-1}\hat{\Pi}_z(k)}{[1 - \hat{D}_L(k)] - |\hat{\Pi}_z(0) - \hat{\Pi}_z(k)\,\hat{D}_L(k)|}
\end{aligned} \tag{3.10}$$

In the last step we used the fact that $g_z \geqslant 1$ and $z \geqslant z_L$. Now, by the triangle inequality, the fact that $|\hat{D}_L(k)| \leqslant 1$, (3.6), and (3.7),

$$\begin{aligned}
|\hat{\Pi}_z(0) - \hat{\Pi}_z(k)\,\hat{D}_L(k)| &\leqslant |\hat{\Pi}_z(0)| \cdot [1 - \hat{D}_L(k)] + |\hat{\Pi}_z(0) - \hat{\Pi}_z(k)| \\
&\leqslant c_2 L^{1-d}[1 - \hat{D}_L(k)] + c_4 L^{3-d} k^2
\end{aligned}$$

By (A.1)

$$k^2 \leqslant c_5[1 - \hat{D}_L(k)] \tag{3.11}$$

and hence by (3.6) and (3.10), if L is sufficiently large, then

$$0 \leqslant \hat{G}_z(k) \leqslant \frac{4}{1 - \hat{D}_L(k)} \tag{3.12}$$

Combining (3.11) and (3.12) gives the infrared bound

$$0 \leqslant \hat{G}_z(k) \leqslant \frac{4c_5}{k^2}$$

By (3.6), (3.9), and the fact that $g_z \geqslant 1$,

$$z\,|\Omega_L| \leqslant g_z \cdot z\,|\Omega_L| < \frac{1}{1 + g_z^{-1}\hat{\Pi}_z(0)} \leqslant 1 + O(L^{1-d}) \leqslant \frac{5}{4} \tag{3.13}$$

for L sufficiently large. Therefore

$$\begin{aligned}|\hat{G}_z(k)-g_z| &= \left|\frac{g_z[1-\hat{F}_z(k)]+\hat{\Pi}_z(k)}{\hat{F}_z(k)}\right| \\ &= \left|\frac{g_z z\,|\Omega_L|\,\hat{D}_L(k)[g_z+\hat{\Pi}_z(k)]+\hat{\Pi}_z(k)}{\hat{F}_z(k)}\right| \\ &\leqslant |\hat{G}_z(k)|\cdot g_z z\,|\Omega_L|\,|\hat{D}_L(k)| + \left|\frac{\hat{\Pi}_z(k)}{\hat{F}_z(k)}\right| \\ &\leqslant 5\,\frac{|\hat{D}_L(k)|+c_2L^{1-d}}{1-\hat{D}_L(k)} \end{aligned} \tag{3.14}$$

using (3.6), (3.12), (3.13), and P_4. We are now ready to obtain the bounds of P_3.

(a) The bound on g_z. By (3.9), (3.6), and the fact that $z > z_L$,

$$g_z < \frac{1}{z\,|\Omega_L|} - \hat{\Pi}_z(0) \leqslant \frac{3e}{(2\pi)^{1/2}} + c_2L^{1-d} \tag{3.15}$$

The right-hand side is less than 4 if L is sufficiently large.

(b) The bound on S. By definition,

$$S = \int \frac{d^dk}{(2\pi)^d}\,[\hat{G}_z(k)-g_z]^2\,\{[\hat{G}_z(k)+g_z]^2+2(g_z)^2\}$$

By (3.12) and the bound (a) on g_z,

$$[\hat{G}_z(k)+g_z]^2+2(g_z)^2 \leqslant 4^2\left[\left(\frac{1}{1-\hat{D}_L(k)}+1\right)^2+2\right]$$

With (3.14) this gives

$$\begin{aligned} S &\leqslant 800\int \frac{d^dk}{(2\pi)^d}\,\frac{\hat{D}_L(k)^2+c_2^2L^{2-2d}}{[1-\hat{D}_L(k)]^2}\left[\left(\frac{1}{1-\hat{D}_L(k)}+1\right)^2+2\right] \\ &= 800[S_G+[C_L(0,0)-1]^2\,\{[C_L(0,0)+1]^2+2\}] + \text{const}\cdot L^{2-2d} \\ &\leqslant 800(K_SL^{1-d}+\text{const}\cdot L^{2-2d}) \end{aligned}$$

using (A.2) and (A.3) in the last two steps. (See also the equation in the proof of Lemma A.1.) This gives $S \leqslant 400\alpha K_S L^{1-d}$ with $\alpha = 3$.

(c) The bound on W. By definition, $W=\sum_{x,y} |x|^2 G_z(0,x)\, G_z(x,y) \times G_z(y,0)$. In terms of the Fourier transform, this can be written (after an integration by parts)

$$W=\sum_{\mu} 2 \int \frac{d^d k}{(2\pi)^d} [\partial_\mu \hat{G}_z(k)]^2\, \hat{G}_z(k) \tag{3.16}$$

Differentiation of (3.8) gives

$$\begin{aligned}[\partial_\mu \hat{G}_z(k)]^2 &= \left[\frac{\partial_\mu \hat{\Pi}_z(k)}{\hat{F}_z(k)} - \frac{g_z+\hat{\Pi}_z(k)}{\hat{F}_z(k)^2}\, \partial_\mu \hat{F}_z(k)\right]^2 \\ &\leqslant 2\,\frac{[\partial_\mu \hat{\Pi}_z(k)]^2}{\hat{F}_z(k)^2} + 2\,\frac{[g_z+\hat{\Pi}_z(k)]^2}{\hat{F}_z(k)^4}\,[\partial_\mu \hat{F}_z(k)]^2 \end{aligned} \tag{3.17}$$

By definition,

$$[\partial_\mu \hat{F}_z(k)]^2 \leqslant 2(z\,|\Omega_L|)^2 \left\{[\partial_\mu \hat{D}_L(k)]^2\, [g_z+\hat{\Pi}_z(k)]^2 + [\partial_\mu \hat{\Pi}_z(k)]^2\right\} \tag{3.18}$$

Since $g_z \geqslant 1$, it follows from (3.6) that for L sufficiently large, $\hat{F}_z(k)^{-1} \leqslant \frac{5}{4}\hat{G}_z(k)$. Using (3.18), it follows from (3.17), (3.12), and (3.13) that

$$[\partial_\mu \hat{G}_z(k)]^2 \leqslant 50\,\frac{[\partial_\mu \hat{\Pi}_z(k)]^2}{[1-\hat{D}_L(k)]^2} + 1600\,\frac{[\partial_\mu \hat{D}_L(k)]^2}{[1-\hat{D}_L(k)]^4} + 2500\,\frac{[\partial_\mu \hat{\Pi}_z(k)]^2}{[1-\hat{D}_L(k)]^4} \tag{3.19}$$

Using (3.19) and (3.12) in (3.16) gives

$$\begin{aligned} W &\leqslant 20{,}000 \sum_{\mu=1}^{d} \int \frac{d^d k}{(2\pi)^d} \left[\frac{[\partial_\mu \hat{\Pi}_z(k)]^2}{[1-\hat{D}_L(k)]^3} + \frac{[\partial_\mu \hat{D}_L(k)]^2}{[1-\hat{D}_L(k)]^5} + \frac{[\partial_\mu \hat{\Pi}_z(k)]^2}{[1-\hat{D}_L(k)]^5}\right] \\ &= 20{,}000 \left[\frac{1}{2} W_G + \sum_{\mu=1}^{d} \int \frac{d^d k}{(2\pi)^d} \left(\frac{[\partial_\mu \hat{\Pi}_z(k)]^2}{[1-\hat{D}_L(k)]^3} + \frac{[\partial_\mu \hat{\Pi}_z(k)]^2}{[1-\hat{D}_L(k)]^5}\right)\right] \end{aligned} \tag{3.20}$$

To bound the final term in (3.20), we note that by symmetry $\partial_\mu \hat{\Pi}_z(k)$ is equal to zero for any k with $k_\mu = 0$. Denoting by $\tilde{k}$ the result of replacing the μth component of k by zero, it follows from Taylor's theorem that

$$\partial_\mu \hat{\Pi}_z(k) = \partial_\mu \hat{\Pi}_z(k) - \partial_\mu \hat{\Pi}_z(\tilde{k}) = k_\mu \partial_\mu^2 \hat{\Pi}_z(k^*) \tag{3.21}$$

where k^* is a point on the line segment joining $\tilde{k}$ and k. By (3.21) and (3.6),

$$|\partial_\mu \hat{\Pi}_z(k)|^2 \leqslant k_\mu^2 c_3^2 L^{6-2d}$$

and hence by symmetry and (3.11), for $d>8$ the last term in (3.20) is $O(L^{6-2d})$ and is negligible compared to the $K_W L^{3-d}$ bound on W_G from (A.4). The other term in the integral in (3.20) is also $O(L^{6-2d})$, by (3.6) and (3.11). This yields the bound of P_3 for W.

(d) The bound on W_l. In terms of the Fourier transform,

$$W_l = -\sum_{\mu=1}^{d} \int \frac{d^d k}{(2\pi)^d} [\partial_\mu^2 \hat{G}_z(k)] \, \hat{G}_z(k)^2 \, e^{ik \cdot l}$$

Differentiation of (3.8) gives

$$\partial_\mu^2 \hat{G}_z(k) = \frac{\partial_\mu^2 \hat{\Pi}_z(k)}{\hat{F}_z(k)} - 2 \frac{\partial_\mu \hat{\Pi}_z(k) \, \partial_\mu \hat{F}_z(k)}{\hat{F}_z(k)^2} - \frac{[g_z + \hat{\Pi}_z(k)] \, \partial_\mu^2 \hat{F}_z(k)}{\hat{F}_z(k)^2} + \frac{2[g_z + \hat{\Pi}_z(k)][\partial_\mu \hat{F}_z(k)]^2}{\hat{F}_z(k)^3} \tag{3.22}$$

Now

$$\partial_\mu \hat{F}_z(k) = -z \, |\Omega_L| \, \{\partial_\mu \hat{D}_L(k)[g_z + \hat{\Pi}_z(k)] + \hat{D}_L(k) \, \partial_\mu \hat{\Pi}_z(k)\}$$

and

$$\partial_\mu^2 \hat{F}_z(k) = -z \, |\Omega_L| \, \{\partial_\mu^2 \hat{D}_L(k)[g_z + \hat{\Pi}_z(k)] + 2\partial_\mu \hat{D}_L(k) \, \partial_\mu \hat{\Pi}_z(k) + \hat{D}_L(k) \, \partial_\mu^2 \hat{\Pi}_z(k)\}$$

Arguing as for W, we have

$$\begin{aligned} W_l &\leqslant c_6 \int \frac{d^d k}{(2\pi)^d} \frac{1}{[1 - \hat{D}_L(k)]^4} \sum_\mu \bigg[|\partial_\mu^2 \hat{D}_L(k)| + O(L^{3-d}) \, |\partial_\mu \hat{D}_L(k)| \\ &\quad + O(L^{3-d}) + \frac{|\partial_\mu \hat{D}_L(k)|^2 + O(L^{6-2d}) \, k_\mu^2}{1 - \hat{D}_L(k)} \bigg] \\ &\leqslant c_7 L^{3-d} + c_6 \sum_\mu \int \frac{d^d k}{(2\pi)^d} \frac{|\partial_\mu^2 \hat{D}_L(k)|}{[1 - \hat{D}_L(k)]^4} + c_6 W_G \\ &\leqslant c_8 L^{3-d} \end{aligned}$$

using (A.4) and (A.5) in the last step.

We take $K' = 4^3 c_8/3$. This proves the bound of P_3 for W_l, and, as mentioned in the discussion following Proposition 3.3, this calculation can also be used to show that W_l satisfies the bound of P_3 if $z \leqslant z_L$. ∎

3.3. Proof that $\nu = 1/4$

We continue to consider only spread-out lattice trees explicitly. Nearest-neighbor trees or animals, and spread-out animals, can be treated similarly. By definition [Eq. (1.9)], in terms of the Fourier transform,

$$\xi(z)^2 = -\sum_{\mu=1}^{d} \frac{\partial_\mu^2 \hat{G}_z(0)}{\hat{G}_z(0)} \tag{3.23}$$

Since $\hat{F}_z(k)^{-1} = [g_z + \hat{\Pi}_z(k)]^{-1} \hat{G}_z(k)$ and since by symmetry $\partial_\mu \hat{F}_z(0) = \partial_\mu \hat{\Pi}_z(0) = 0$, it follows from direct calculation and (3.22) that

$$\begin{aligned}
\xi(z)^2 &= [g_z + \hat{\Pi}_z(0)]^{-1} \hat{G}_z(0) \sum_{\mu=1}^{d} \partial_\mu^2 \hat{F}_z(0) - [g_z + \hat{\Pi}_z(0)]^{-1} \sum_{\mu=1}^{d} \partial_\mu^2 \hat{\Pi}_z(0) \\
&= -z\, |\Omega_L|\, [g_z + \hat{\Pi}_z(0)]^{-1} \hat{G}_z(0) \sum_{\mu=1}^{d} \{\partial_\mu^2 \hat{D}_L(0) [g_z + \hat{\Pi}_z(0)] \\
&\quad + \partial_\mu^2 \hat{\Pi}_z(0)\} - [g_z + \hat{\Pi}_z(0)]^{-1} \sum_{\mu=1}^{d} \partial_\mu^2 \hat{\Pi}_z(0)
\end{aligned}$$

Since $\partial_\mu^2 \hat{D}_L(0) \sim -L^2$ for large L by direct calculation, it follows from (3.6), (3.13), and (3.15) that

$$\xi(z)^2 \sim \hat{G}_z(0) = \chi(z)$$

for z close to z_c. By (1.15), this proves $\nu = 1/4$.

APPENDIX. BOUNDS ON GAUSSIAN QUANTITIES

In this appendix we collect the estimates on Gaussian (simple random walk) quantities that we need to treat the spread-out models. Analogous estimates for the nearest-neighbor models are given in Appendix B.2 of ref. 10.

We begin by recalling the definitions

$$\hat{D}_L(k) = \frac{1}{|\Omega_L|} \sum_{x \in (\mathbf{Z}^d \cap \Omega_L) \setminus \{0\}} e^{ik \cdot x}$$

$$C_L(x, y) = \int_{[-\pi, \pi]^d} \frac{d^d k}{(2\pi)^d} \frac{e^{ik \cdot (y-x)}}{1 - \hat{D}_L(k)}$$

$$S_G = \sum_{x, y, w} C_L(0, x)\, C_L(x, y)\, C_L(y, w)\, C_L(w, 0) - C_L(0, 0)^4$$

$$W_G = \sum_{x, y} |x|^2\, C_L(0, x)\, C_L(x, y)\, C_L(y, 0)$$

The estimates we need are given in the following lemma. (The lemma can likely be improved to the corresponding inequalities with $\varepsilon = 0$, but these suffice for our purposes.)

Lemma A.1. Fix $d > 8$. Then there are constants K_S, K_W, c_5, c, and $\varepsilon \leqslant 1/5$ such that for all $L \geqslant 1$,

$$k^2 \leqslant c_5[1 - \hat{D}_L(k)] \tag{A.1}$$

$$0 \leqslant C_L(0, 0) - 1 \leqslant cL^{-d+\varepsilon} \tag{A.2}$$

$$S_G \leqslant K_S L^{-d+4\varepsilon} \tag{A.3}$$

$$W_G \leqslant K_W L^{-d+2+5\varepsilon} \tag{A.4}$$

$$\int \frac{d^dk}{(2\pi)^d} \frac{|\partial_\mu^2 \hat{D}_L(k)|}{[1 - \hat{D}_L(k)]^4} \leqslant \text{const} \cdot L^{-d+2+4\varepsilon} \tag{A.5}$$

Proof. The bound (A.1) is proved in ref. 6, Lemma 5.1. The bound (A.2) is proved in ref. 6, Corollary 5.8, and (A.3) follows from Lemma 5.7 and the proof of Corollary 5.8 of ref. 6, using the representation

$$S_G = \int \frac{d^dk}{(2\pi)^d} [\hat{C}_L(k) - 1]^2 \{[\hat{C}_L(k) + 1]^2 + 2\}$$
$$- [C_L(0, 0) - 1]^2 \{[C_L(0, 0) + 1]^2 + 2\}$$

Finally, (A.4) and (A.5) can be proved just as in Lemmas 5.10 and 5.11 of ref. 6. ∎

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